Lie Theory and Physics

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Preface

Sophus Lie (1842-1899) was a Norwegian mathematician, who created an *algebraic* language (Lie algebras) to deal with the notion of symmetry in the *analytic* setting (Lie groups).

1 Constructions of linear algebra

Suppose U, V are two finite dimensional complex vector spaces. The set theoretic direct product $U \times V$ has a natural vector space structure by componentwise addition and scalar multiplication, so

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2),\ \lambda(u, v) := (\lambda u, \lambda v)$$

for all $u_1, u_2, u \in U,\ v_1, v_2, v \in V$ and $\lambda \in \mathbb{C}$. This vector space is denoted $U \oplus V$ and is called the *direct sum* of U and V. It is easy to check that $\dim(U \oplus V) = \dim U + \dim V$.

The set of all linear maps $A : U \to V$ has a natural vector space structure under pointwise addition and scalar multiplication, so

$$(A_1 + A_2)(u) := A_1 u + A_2 u,\ (\lambda A)(u) = \lambda(Au)$$

for all linear maps $A_1, A_2, A : U \to V$ and all $u \in U$ and $\lambda \in \mathbb{C}$. We denote this vector space $\mathrm{Hom}(U, V)$. It is easy to check that $\dim \mathrm{Hom}(U, V) = \dim U \dim V$. We also denote $\mathrm{Hom}(U, U)$ by $\mathrm{End}(U)$. The vector space $\mathrm{Hom}(U, \mathbb{C})$ is also denoted U^* and called the dual vector space of U.

Suppose U, V, W are three finite dimensional complex vector spaces. A map $b : U \times V \to W$ is called *bilinear* if for fixed $v \in V$ the map $U \to W,\ u \mapsto b(u, v)$ is linear, and likewise for fixed $u \in U$ the map $V \to W,\ v \mapsto b(u, v)$ is linear. The *tensor product* of the vector spaces U and V is a vector space $U \otimes V$ together with a bilinear map $i : U \times V \to U \otimes V$, also denoted $i(u, v) = u \otimes v$ for $u \in U$ and $v \in V$, such that for each bilinear map $b : U \times V \to W$ to a third vector space W there exists a unique linear map $B : U \otimes V \to W$ with $b(u, v) = B(u \otimes v)$ for all $u \in U$ and $v \in V$. The linear map B is called the *lift* of the bilinear map b (from the direct product $U \times V$ to the tensor product $U \otimes V$). The uniqueness of the linear map B implies that the cone $\{u \otimes v; u \in U, v \in V\}$ of pure tensors in $U \otimes V$ spans the vector space $U \otimes V$. Here we use the word *cone* for a subset of a vector space, that is invariant under scalar multiplication.

The tensor product is unique up to natural isomorphism. Indeed, if $U \boxtimes V$ is another tensor product with associated bilinear map $j : U \times V \to U \boxtimes V$ and $j(u, v) = u \boxtimes v$ then there exist unique linear lifts

$$J : U \otimes V \to U \boxtimes V,\ J(u \otimes v) = u \boxtimes v$$

of $j : U \times V \to U \boxtimes V$ and

$$I : U \boxtimes V \to U \otimes V,\ I(u \boxtimes v) = u \otimes v$$

of $i : U \times V \to U \otimes V$, and so $IJ = \mathrm{id}_{U \otimes V}$ and $JI = \mathrm{id}_{U \boxtimes V}$. Hence I and J are inverses of each other.

For the existence of the tensor product we can take for $U \otimes V$ the free vector space $F(U \times V)$ on the set $U \times V$ modulo the linear subspace of $F(U \times V)$ spanned by the elements

$$(u_1 + u_2, v) - (u_1, v) - (u_2, v),\ (\lambda u, v) - \lambda(u, v)$$
$$(u, v_1 + v_2) - (u, v_1) - (u, v_2),\ (u, \lambda v) - \lambda(u, v)$$

for all $u_1, u_2, u \in U$, $v_1, v_2, v \in V$ and $\lambda \in \mathbb{C}$.

If $A \in \mathrm{End}(U)$ and $B \in \mathrm{End}(V)$ then $A \otimes B \in \mathrm{End}(U \otimes V)$ is defined by $(A \otimes B)(u \otimes v) = A(u) \otimes B(v)$ for all $u \in U$ and $v \in V$. If $A_1, A_2 \in \mathrm{End}(U)$ and $B_1, B_2 \in \mathrm{End}(V)$ then clearly $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$.

The bilinear map $U^* \times V \to \mathrm{Hom}(U, V)$ sending the pair (f, v) to the linear map $u \mapsto f(u)v$ lifts to a linear map $U \otimes V \to \mathrm{Hom}(U, V)$ sending the pure tensor $f \otimes v$ to the linear map $u \mapsto f(u)v$. It is easy to see that this linear map is a linear isomorphism, and so we have a natural isomorphism $U^* \otimes V \cong \mathrm{Hom}(U, V)$.

The tensor product of vector spaces is associative in the sense that the linear spaces $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are naturally isomorphic. Likewise $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ on $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ for $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$ and $C \in \mathrm{End}(W)$. The *tensor algebra* $\mathrm{T}V$ on the vector space V is defined as the graded direct sum $\oplus \mathrm{T}^k V$ with $\mathrm{T}^k V$ inductively defined by $\mathrm{T}^0 V = \mathbb{C}$ and $\mathrm{T}^{k+1} V = \mathrm{T}^k V \otimes V$ for all $k \in \mathbb{N}$. By associativety we have $\mathrm{T}^k V \otimes \mathrm{T}^l V \cong \mathrm{T}^{k+l} V$ for all $k, l \in \mathbb{N}$, and so $\mathrm{T}V = \oplus \mathrm{T}^k V$ becomes a graded associative algebra with respect to the tensor product map $\otimes$ as multiplication.

There are two graded ideals $\mathrm{I}_\pm V = \oplus \mathrm{I}^k_\pm V$ of $\mathrm{T}V$ generated (through left and right multiplications) by the degree two tensors $(v_1 \otimes v_2 \pm v_2 \otimes v_1)$ for all $v_1, v_2 \in V$. The quotient algebras $\mathrm{S}V = \oplus \mathrm{S}^k V = \mathrm{T}V/\mathrm{I}_- V$ and $\wedge V = \oplus \wedge^k V = \mathrm{T}V/\mathrm{I}_+ V$ are the so called *symmetric algebra* and *exterior algebra* on V respectively. The symmetric algebra $\mathrm{S}V$ is commutative for the induced multiplication, and this multiplication is usually suppressed in the notation (like with multiplication in a group). The induced multiplication on the exterior algebra $\wedge V$ is denoted by $\wedge$ and called the *wedge product*. We have $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for $\alpha \in \wedge^k V$ and $\beta \in \wedge^l V$.

In case $V = U^*$ is the dual space of a vector space U then $\mathrm{T}^k V$ can be identified with the space of scalar valued multilinear forms in k arguments from U. Here *multilinear* is used in the sense that, if all arguments except one remain fixed, then the resulting form is linear in that one argument. In turn $\mathrm{S}V = \mathrm{P}U = \oplus \mathrm{P}^k U$ can be identified with the space of polynomial functions on U by restriction to the main diagonal in U^k. Likewise $\wedge^k V = \wedge^k U^*$ can be identified with the space of alternating multilinear forms in k arguments from U. So $\alpha \in \wedge^k U^*$ means that α is multilinear in k arguments from U and $\alpha(u_{\sigma(1)}, \cdots, u_{\sigma(k)}) = \varepsilon(\sigma)\alpha(u_1, \cdots, u_k)$ for σ in the symmetric group $\mathfrak{S}_k$.

Exercise 1.1. *Show that* $(U \otimes V) \otimes W$ *and* $U \otimes (V \otimes W)$ *are naturally isomorphic for* U, V, W *vector spaces.*

Exercise 1.2. *Show that for* $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$ *and so* $A \otimes B \in \mathrm{End}(U \otimes V)$ *we have* $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$.

Exercise 1.3. *Show that for a vector space* V *of dimension* n *one has*

$$\dim \mathrm{S}^k V = \binom{n+k-1}{k}, \ \dim \wedge^k V = \binom{n}{k}$$

for all $k \in \mathbb{N}$.

2 Representations of groups

Throughout this section let G be a group and U, V, W finite dimensional vector spaces over the complex numbers.

A *representation* of a group G on a vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Note that necessarily V is nonzero. So a representation of G will be given by a pair (ρ, V) with V a finite dimensional vector space, called the representation space, and $\rho : G \to \mathrm{GL}(V)$ a homomorphism. If $\rho(x) = 1$ for all $x \in G$ then we speak of the *trivial* representation of G on V. A linear subspace U of a representation space V of G is called *invariant* if for all $x \in G$ and all $u \in U$ we have $\rho(x)u \in U$. If in addition U is nonzero then we get a subrepresentation (ρ_U, U) of (ρ, V) defined by $\rho_U(x) = \rho(x)|_U$ for all $x \in G$. Likewise, if U is a proper invariant subspace of V then we get a quotient representation $(\rho_{V/U}, V/U)$ defined by $\rho_{V/U}(x)(v+U) = \rho(x)(v)+U$ for $x \in G$ and $v \in V$.

We say that a representation (ρ, V) of G is *irreducible* if the only two invariant linear subspaces of V are the trivial invariant linear subspaces $\{0\}$ and V. For example, the trivial representation of G on V is irreducible if and only if $\dim V = 1$. If (ρ, V) is a representation of a group G and $H < G$ is a subgroup then by restriction (ρ, V) becomes also a representation of H. This process is called *branching* or *symmetry breaking* from G to H.

Using constructions of linear algebra one can make new representations from old ones. For example, the *outer tensor product* of two representations (ρ_1, V_1) of a group G_1 and (ρ_2, V_2) of a group G_2. The outer tensor product $\rho_1 \boxtimes \rho_2$ is a representation of the direct product group $G_1 \times G_2$ on the vector space $V_1 \otimes V_2$, and is defined by $(\rho_1 \boxtimes \rho_2)(x_1, x_2) = \rho_1(x_1) \otimes \rho_2(x_2)$ for $x_1 \in G_1$ and $x_2 \in G_2$. If $G_1 = G_2 = G$ then the *inner tensor product* $\rho_1 \otimes \rho_2$ of two representations (ρ_1, V_1) and (ρ_2, V_2) of the same group G is a representation of that group G on $V_1 \otimes V_2$, which is defined by $(\rho_1 \otimes \rho_2)(x) = (\rho_1(x)) \otimes (\rho_2(x))$ for $x \in G$. So the inner tensor product is obtained from the outer tensor product by branching from the direct product group $G \times G$ to the diagonal subgroup G.

If (ρ, V) is a representation of a group G then we also get representations $(\mathrm{S}^k\rho, \mathrm{S}^k V)$ and $(\wedge^k \rho, \wedge^k V)$ of the same group G, called the symmetric power and the wedge power representations of degree k respectively.

If $A \in \mathrm{Hom}(U, V)$ then the dual linear map $A^* \in \mathrm{Hom}(V^*, U^*)$ will be defined by $(A^* f)u = f(Au)$ for $f \in V^*$ and $u \in U$. It is easy to check that $(BA)^* = A^* B^*$ for $A \in \mathrm{Hom}(U, V)$ and $B \in \mathrm{Hom}(V, W)$. If (ρ, V)

is a representation of a group G then the *dual representation* (ρ^*, V^*) of G is defined by $\rho^*(x) = (\rho(x^{-1}))^*$ for $x \in G$. More generally, if (ρ, V) and (σ, W) are two representations of a group G then the representation $(\mathrm{Hom}(\rho,\sigma), \mathrm{Hom}(V,W))$ of G is defined by $\mathrm{Hom}(\rho,\sigma)(x)A = \sigma(x)A\rho(x^{-1})$ for $x \in G$ and $A \in \mathrm{Hom}(V,W)$.

If (ρ, V) is a representation of a group G then we denote by V^G the linear subspace $\{v \in V; \rho(x)v = v \ \forall x \in G\}$ of V of fixed vectors. For example, for representations (ρ, V) and (σ, W) of G the space $\mathrm{Hom}(V,W)^G$ consists of those $A \in \mathrm{Hom}(V,W)$ for which $A\rho(x) = \sigma(x)A$ for all $x \in G$. Such a linear map $A \in \mathrm{Hom}(V,W)$ is called an *intertwining operator* or simply an *intertwiner* from (ρ, V) to (σ, W). The two representations (ρ, V) and (σ, W) of G are called *equivalent* if there exists an intertwiner $\mathrm{Hom}(V,W)^G$ which is also a linear isomorphism, and we write $(\rho, V) \sim (\sigma, W)$. The natural isomorphism $\mathrm{Hom}(V,W) \cong V^* \otimes W$ gives an equivalence $\mathrm{Hom}(\rho,\sigma) \sim \rho^* \otimes \sigma$ of representations.

However, in daily life people are sometimes sloppy and use the word representation while equivalence class of representation is meant. Be aware of that.

A representation (ρ, V) of a group G is called *completely reducible* if for any invariant subspace U of V there exists a complementary invariant subspace $U^\perp$. So $U + U^\perp = V$ and $U \cap U^\perp = \{0\}$, which is also denoted $V = U \oplus U^\perp$, and moreover $U^\perp$ is invariant as well. We denote $\rho = \rho_U \oplus \rho_{U^\perp}$ for this situation (strictly speaking if $U, U^\perp$ are both nonzero).

Suppose the vector space V carries a Hermitian inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$, which by definition is a positive definite sesquilinear form on V, usually linear in the first argument and antilinear in the second argument, and interchange of arguments amounts to complex conjugation. The pair $(V, \langle\cdot,\cdot\rangle)$ is called a finite dimensional *Hilbert space*. A representation (ρ, V) of a group G on a Hilbert space $(V, \langle\cdot,\cdot\rangle)$ is called *unitary* if

$$\langle \rho(x)v_1, \rho(x)v_2 \rangle = \langle v_1, v_2 \rangle$$

for all $x \in G$. So a unitary representation is a triple $(\rho, V, \langle\cdot,\cdot\rangle)$ with $(V, \langle\cdot,\cdot\rangle)$ a Hilbert space and $\rho : G \to \mathrm{U}(V, \langle\cdot,\cdot\rangle)$ a homomorphism of G into the *unitary group* of that Hilbert space.

Lemma 2.1. *A unitary representation $(\rho, V, \langle\cdot,\cdot\rangle)$ of a group G is always completely reducible.*

Proof. If $U \subset V$ is an invariant subspace then the orthogonal complement $U^\perp = \{v \in V; \langle u, v\rangle = 0 \; \forall \; u \in U\}$ is easily seen to be also invariant. □

Theorem 2.2. *Any representation (ρ, V) of a* finite *group G is completely reducible.*

Proof. We claim that the representation (ρ, V) is always *unitarizable*, that is can be made unitary for some Hermitian inner product $\langle \cdot, \cdot \rangle$ on V. Just pick any Hermitian inner product $\langle \cdot, \cdot \rangle'$ on V. By averaging over G we obtain a new Hermitian inner product

$$\langle v_1, v_2 \rangle = \frac{1}{|G|} \sum_{x \in G} \langle \rho(x) v_1, \rho(x) v_2 \rangle'$$

and it is easily checked that $\langle \rho(x) v_1, \rho(x) v_2 \rangle = \langle v_1, v_2 \rangle$ for all $x \in G$. So $(\rho, V, \langle \cdot, \cdot \rangle)$ becomes a unitary representation, and we can apply the above lemma. □

Remark 2.3. *The above theorem has an extension to compact topological groups and continuous finite dimensional representations of such groups. A* topological *group G is both a topological space and a group, and the two structures are compatible in the sense that multiplication $G \times G \to G, (x, y) \mapsto xy$ and inversion $G \to G, x \mapsto x^{-1}$ are continuous maps. Finite groups are compact topological groups relative to the discrete topology, and so any representation of a finite group is automatically continuous. The simplest example of a compact topological group, which is not a finite group, is the circle group $\mathbb{R}/2\pi\mathbb{Z}$. Geometrically $\mathbb{R}/2\pi\mathbb{Z} \cong \mathrm{U}_1(\mathbb{C}) \cong \mathrm{SO}_2(\mathrm{R})$ is the group of rotations of the complex plane $\mathbb{C} \cong \mathbb{R}^2$ around the origin.*

It is a nontrivial theorem of John von Neumann (1903-1957) that for any compact topological group G there exists a positive measure *μ that is* invariant under left and right multiplications. *Such a measure μ on G assigns to a continuous function $f : G \to \mathbb{C}$ by integration against μ a complex number*

$$\int_G f(x) d\mu(x)$$

which is positive if f is positive and satisfies the invariance relations

$$\int_G f(xy) d\mu(x) = \int_G f(yx) d\mu(x) = \int_G f(x) d\mu(x)$$

for all $y \in G$. Moreover, such a measure becomes unique by the normalization $\int_G d\mu(x) = 1$. For the circle group $\mathbb{R}/2\pi\mathbb{Z}$ with standard coordinate θ this normalized invariant measure is just

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

the usual Riemann integral. Now the proof of the above theorem carries verbatim over to the present situation by using

$$\langle v_1, v_2\rangle = \int_G \langle \rho(x)v_1, \rho(x)v_2\rangle' d\mu(x)$$

in stead of the finite normalized sum as above.

The conclusion is that the representation theory of finite (or even compact) groups boils down to two fundamental problems. In the first place classify the equivalence classes of irreducible representations and preferably give constructions and nice models for each of them. In the second place decompose a given representation into irreducible ones.

An easy but important result is Schur's lemma, which was found by Issai Schur (1875-1941) in 1905.

Lemma 2.4. *If (ρ, U) is an irreducible representation of a group G then the vector space $\mathrm{End}(U)^G$ of self intertwiners for (ρ, U) consists only of scalar linear maps. If (ρ, U) and (σ, V) are both irreducible representations of G then either $\dim \mathrm{Hom}(U, V)^G$ is equal to 1 if $(\rho, U) \sim (\sigma, V)$ or is equal to 0 if $(\rho, U) \nsim (\sigma, V)$.*

Proof. Suppose $A \in \mathrm{End}(U)^G$ is a self intertwiner for (ρ, U). Since scalars are self intertwiners we also have $(A - \lambda) \in \mathrm{End}(U)^G$ for all $\lambda \in \mathbb{C}$. If λ is an eigenvalue of A then the eigenspace $\ker(A - \lambda)$ is a nonzero invariant subspace of U and so equal to U itself by irreducibility of (ρ, U). Hence $A = \lambda$ for some $\lambda \in \mathbb{C}$ and the first statement follows.

Suppose $A \in \mathrm{Hom}(U, V)^G$ is a nonzero intertwiner. Then $\ker A$ is a proper invariant subspace of U and so is equal to $\{0\}$ since (ρ, U) is irreducible. Also $\mathrm{im}\, A$ is a nonzero invariant subspace of V and so is equal to V since (σ, V) is irreducible. Hence any nonzero intertwiner $A \in \mathrm{Hom}(U, V)^G$ is a linear isomorphism and so gives an equivalence $(\rho, U) \sim (\sigma, V)$ of representations.

If $A, B \in \mathrm{Hom}(U, V)^G$ with $A \neq 0$ then $BA^{-1} \in \mathrm{End}(V)^G$ is a self intertwiner for (σ, V). Hence $BA^{-1} = \lambda$ for some $\lambda \in \mathbb{C}$ and so $B = \lambda A$, which implies $\dim \operatorname{Hom}(U,V)^G = 1$. On the other hand, if $(\rho, U) \nsim (\sigma, V)$ then clearly $\dim \operatorname{Hom}(U,V)^G = 0$. $\square$

Corollary 2.5. *An irreducible representation of an Abelian group is one dimensional.*

Proof. Suppose (ρ, V) is an irreducible representation of an Abelian group G. Then all linear operators $\rho(x)$ for $x \in G$ are self intertwiners, and hence scalars by Schur's lemma. Hence any linear subspace of V is invariant and therefore $\dim V = 1$ by the definition or irreducibility. $\square$

Exercise 2.1. *Check that a positive linear combination of Hermitian inner products on a vector space V is again a Hermitian inner product.*

Exercise 2.2. *Let B_n be the subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$ of upper triangular matrices, so $A = (a_{ij}) \in B_n$ if and only if $a_{ij} = 0$ for $i > j$. Show that the natural representation of B_n on $\mathbb{C}^n$ is not completely reducible for $n \geq 2$.*

Exercise 2.3. *In this section all vector spaces were finite dimensional and over the complex numbers $\mathbb{C}$. Show that Schur's lemma does not hold over the real numbers $\mathbb{R}$ by looking at the real representation of the circle group*

$$\mathbb{R}/2\pi\mathbb{Z} \to \mathrm{GL}_2(\mathbb{R}), \theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

by rotations of the real plane $\mathbb{R}^2$.

3 Character theory for finite groups

If V is a finite dimensional vector space with basis e_i then the *trace* $\operatorname{tr} A$ of a linear map $A : V \to V$, with matrix (a_{ij}) given by $Ae_j = \sum_i a_{ij} e_i$, is defined by $\operatorname{tr} A = \sum_i a_{ii}$. One easily checks that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A, B \in \operatorname{End}(V)$ and in particular $\operatorname{tr}(BAB^{-1}) = \operatorname{tr} A$ in case $\det B \neq 0$. In turn this implies that $\operatorname{tr} A$ is well defined, independent of the chosen basis.

Suppose G is a group and (ρ, V) a finite dimensional representation. The *character* $\chi_\rho : G \to \mathbb{C}$ of (ρ, V) is defined by

$$\chi_\rho(x) = \operatorname{tr} \rho(x)$$

for $x \in G$. For finite groups we will show that the character of a finite dimensional representation characterizes the representation upto equivalence, which explains the terminology character.

We list some simple properties of characters. Suppose (ρ, V) and (σ, W) are both representations of G. Clearly $\chi_\rho = \chi_\sigma$ if $(\rho, V) \sim (\sigma, W)$. It is also clear that $\chi_\rho(yxy^{-1}) = \chi_\rho(x)$ for all $x, y \in G$ and so characters are constant on conjugation classes of G. It is also obvious that $\chi_\rho(e) = \dim V$. Finally

$$\chi_{\rho\oplus\sigma}(x) = \chi_\rho(x) + \chi_\sigma(x), \chi_{\rho\otimes\sigma}(x) = \chi_\rho(x)\chi_\sigma(x)$$

and (the second equality sign only for G a finite group)

$$\chi_{\rho^*}(x) = \chi_\rho(x^{-1}) = \overline{\chi_\rho(x)}$$

for all $x \in G$.

From now on we shall assume that G is a *finite* group. Let $\mathrm{L}(G)$ denote the vector space of complex valued functions on G. For $\phi, \psi \in \mathrm{L}(G)$ we define an Hermitian inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{x\in G} \phi(x)\overline{\psi(x)}$$

turning $\mathrm{L}(G)$ into a finite dimensional Hilbert space. The linear subspace of class functions in $\mathrm{L}(G)$, that is functions constant on conjugation classes, is denoted $\mathrm{C}(G)$. So characters of representations of G are elements of $\mathrm{C}(G)$.

The next theorem states the so called Schur orthogonality relations for the irreducible characters of G. The proof is a beautiful application of Schur's lemma, and was in fact the principal motivation for Schur to come up with his lemma.

Theorem 3.1. *If (ρ, V) and (σ, W) are irreducible representations of a finite group G then*

$$\langle \chi_\rho, \chi_\sigma \rangle = \delta_{\rho\sigma}$$

with the Kronecker delta notation $\delta_{\rho\sigma} = 1$ if $\rho \sim \sigma$ and $\delta_{\rho\sigma} = 0$ if $\rho \nsim \sigma$.

Proof. Let (τ, U) be an arbitrary representation of G, and let U^G be the space of fixed vectors for G in U. Define the linear map

$$P : U \to U, Pu = \frac{1}{|G|} \sum_{x \in G} \tau(x)u$$

for $u \in U$. Then $\tau(x)P = P$ for all $x \in G$ and hence $P^2 = P$. So P is a linear projection operator with image $\operatorname{im} P = U^G$. In turn this implies

$$\langle \chi_\tau, 1 \rangle = \frac{1}{|G|} \sum_{x \in G} \operatorname{tr}(\tau(x)) = \operatorname{tr}(P) = \dim U^G$$

as the trace is a linear form and the trace of a linear projection operator is the dimension of its image.

If (ρ, V) and (σ, W) are irreducible representations of G then we shall apply the above formula to the representation $\tau = \operatorname{Hom}(\rho, \sigma) \sim \rho^* \otimes \sigma$ on the vector space $U = \operatorname{Hom}(V, W) \cong V^* \otimes W$ as defined by $\tau(x)A = \sigma(x)A\rho(x^{-1})$ for $x \in G$ and $A \in \operatorname{Hom}(V, W)$. Since by Schur's lemma the space of intertwiners $U^G = \operatorname{Hom}(V, W)^G$ has dimension $\delta_{\rho\sigma}$ we get

$$\langle \chi_\sigma, \chi_\rho \rangle = \langle \chi_{\rho^*} \chi_\sigma, 1 \rangle = \langle \chi_{\rho^* \otimes \sigma}, 1 \rangle = \langle \chi_{\operatorname{Hom}(\rho,\sigma)}, 1 \rangle = \delta_{\rho\sigma}$$

and the Schur orthogonality relations follow from $\langle \chi_\rho, \chi_\sigma \rangle = \overline{\delta_{\rho\sigma}} = \delta_{\rho\sigma}$. $\square$

Suppose (ρ, V) is a representation of G, which by complete reducibility can be written $(\rho_1 \oplus \cdots \oplus, \rho_m, V_1 \oplus \cdots \oplus V_m)$ as a direct sum of irreducibles. Let (σ, W) be an irreducible representation of G. Then the number of irreducible components (ρ_i, V_i) in (ρ, V) equivalent to (σ, W) is equal to $\langle \chi_\rho, \chi_\sigma \rangle \in \mathbb{N}$ and is called the *multiplicity* of (σ, W) in (ρ, V). The characters of the irreducible representations of G are called the irreducible characters of G.

Corollary 3.2. *Two representations of a finite group G with equal characters are equivalent, and so the character characterizes a representation upto equivalence.*

Remark 3.3. *The Schur orthogonality relations have a natural extension from finite to compact groups. Indeed, let G be a compact group with μ the normalized invariant measure on the space $\mathrm{L}(G)$ of continuous functions on G. If (ρ, V) and (σ, W) are irreducible representations of G then we have*

$$\langle \chi_\rho, \chi_\sigma \rangle = \delta_{\rho\sigma}$$

relative to the Hermitian inner product

$$\langle \phi, \psi \rangle = \int_G \phi(x)\overline{\psi(x)} d\mu(x)$$

of $\phi, \psi \in \mathrm{L}(G)$.

For example, the circle group $\mathbb{R}/2\pi\mathbb{Z}$ has irreducible representations

$$\rho_n : \mathbb{R}/2\pi\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times, \rho_n(\theta) = e^{in\theta}$$

indexed by $n \in \mathbb{Z}$. The Schur orthogonality relations amount to

$$\frac{1}{2\pi} \int_0^{2\pi} \rho_m(\theta)\overline{\rho_n(\theta)} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \delta_{mn}$$

for $m, n \in \mathbb{Z}$ which are familiar from Fourier analysis. It is known from Fourier analysis that the space of Fourier polynomials is dense in $\mathrm{L}^2(\mathbb{R}/2\pi\mathbb{Z})$ which in turn implies that each irreducible representation of $\mathbb{R}/2\pi\mathbb{Z}$ is equal to ρ_n for some $n \in \mathbb{Z}$.

As before let us assume that G is a finite group. If $\phi \in \mathrm{L}(G)$ and (ρ, V) is a representation of G then the linear operator

$$\rho(\phi) = \frac{1}{|G|} \sum_{x \in G} \phi(x)\rho(x) \in \mathrm{End}(V)$$

has trace equal to $\mathrm{tr}\rho(\phi) = \langle \phi, \chi_{\rho^*} \rangle$. If $\phi \in \mathrm{C}(G)$ is a class function on G then it is easy to check that $\rho(\phi) \in \mathrm{End}(V)^G$ is an intertwiner. If in addition (ρ, V) is irreducible then $\rho(\phi)$ is a scalar by Schur's lemma. It follows by taking traces that this scalar is equal to $\langle \phi, \chi_{\rho^*} \rangle / \dim V$. Hence if $\phi \in \mathrm{C}(G)$ is orthogonal to all irreducible characters of G then by complete reducibility $\rho(\phi) = 0$ for any representation (ρ, V) of G.

Theorem 3.4. *The irreducible characters form an orthonormal basis of $\mathrm{C}(G)$ with respect to the standard Hermitian inner product $\langle \cdot, \cdot \rangle$.*

Proof. The left regular representation λ of G on the vector space $\mathrm{L}(G)$ is defined by $(\lambda(x)\phi)(y) = \phi(x^{-1}y)$ for $x, y \in G$. The functions $\delta_x \in \mathrm{L}(G)$ for $x \in G$ are defined by $\delta_x(y) = \delta_{xy}$ for $x, y \in G$ and form a basis of $\mathrm{L}(G)$ since $\phi = \sum_x \phi(x)\delta_x$ for all $\phi \in \mathrm{L}(G)$. Since $\lambda(x)\delta_y = \delta_{xy}$ for $x, y \in G$ we get $\phi = \sum_x \phi(x)\lambda(x)\delta_e = |G|\lambda(\phi)\delta_e$ for $\phi \in \mathrm{L}(G)$.

Let $\phi \in \mathrm{C}(G)$ be a class function which is orthogonal to all irreducible characters of G. Then $\rho(\phi) = 0$ for any representation (ρ, V) of G. Taking (ρ, V) equal to the left regular representation $(\lambda, \mathrm{L}(G))$ we conclude that $\lambda(\phi) = 0$. Hence $\phi = |G|\lambda(\phi)\delta_e = 0$ and so the result follows. $\square$

The dimension of the space $\mathrm{C}(G)$ of class functions is equal to the number of conjugation classes in G. Hence the number of equivalence classes of irreducible representations of G is equal to the number of conjugation classes in G.

Example 3.5. *The four group (in German Vierergruppe)* V_4 *of Klein is a group with a unit element* e *and three involutions* a, b, c *which in turn implies* $ab = ba = c, bc = cb = a, ca = ac = b$. *The character table is a matrix of the form*

V_4	e	a	b	c
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

where the rows are indexed by the irreducible characters and the columns are indexed by representatives of the conjugation classes. The matrix element is the value of the irreducible character at the conjugation class.

Example 3.6. *The symmetric group* $\mathfrak{S}_4$ *on* 4 *letters has character table*

$\mathfrak{S}_4$	e	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

with χ_1 *the trivial character,* χ_2 *the sign character* ε, χ_3 *the lift from the two dimensional irreducible character of* $\mathfrak{S}_3$ *via the isomorphism* $\mathfrak{S}_4/\mathrm{V}_4 \cong \mathfrak{S}_3$,

χ_4 the character of the representation ρ_4 via reflections and rotations of the tetrahedron and $\chi_5 = \chi_2\chi_4$ the character of the representation $\rho_5 = \varepsilon \otimes \rho_4$ of $\mathfrak{S}_4$. In order to do Hermitian inner product calculations with characters one should check that the conjugation classes have $1, 6, 3, 8, 6$ elements respectively.

Example 3.7. *The alternating group $\mathfrak{A}_4$ on 4 letters has 4 conjugation classes with representatives the unit e, $a = (12)(34)$, $b = (123)$, $c = (132)$ with $1, 3, 4, 4$ elements respectively. The character table of $\mathfrak{A}_4$ becomes*

$\mathfrak{A}_4$	e	a	b	c
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

with ω a nontrivial cube root of unity. The characters χ_i for $i = 1, 2, 3$ are pull backs from the irreducible charcters of $\mathfrak{A}_4/\mathrm{V}_4$, which is cyclic of order 3. The remaining irreducible character χ_4 is just the restriction to $\mathfrak{A}_4$ of the character χ_4 of $\mathfrak{S}_4$.

Example 3.8. *The alternating group $\mathfrak{A}_5$ on 5 letters has 5 conjugation classes with representatives the unit e, $a = (12)(34)$, $b = (123)$, $c = (12345)$, $d = (13524)$ and with $1, 15, 20, 12, 12$ elements respectively. The icosahedron has 12 vertices, 30 edges and 20 faces in accordance with Euler's formula $12 - 30 + 20 = 2$. The 15 lines through midpoints of opposite edges fall apart in 5 orthogonal triples, giving an isomorphism from the rotation group of the icosahedron onto $\mathfrak{A}_5$. The class of a consists of order 2 rotations around any of these 15 lines. The class of b consists of order 3 rotations, 2 around any line through midpoints of opposite faces, so altogether 20. The rotations of order 5 around the lines through opposite vertices fall into 2 classes of equal size, namely 2 rotations over $2\pi/5$ and 2 rotations over $4\pi/5$ for each of these 6 lines. The character table of $\mathfrak{A}_5$ becomes*

$\mathfrak{A}_5$	e	a	b	c	d
χ_1	1	1	1	1	1
χ_2	3	-1	0	τ	τ'
χ_3	3	-1	0	τ'	τ
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

with $\tau = (1+\sqrt{5})/2$ the golden ratio and $\tau' = (1-\sqrt{5})/2 = -1/\tau$. These two numbers are the roots of the equation $x^2 - x - 1 = 0$. The representation ρ_2 via rotations of the icosahedron has character χ_2 and ρ_3 is its Galois conjugate. The representation ρ_4 with character χ_4 is just the restriction of the reflection representation from $\mathfrak{S}_5$ to $\mathfrak{A}_5$ while $\chi_2\chi_3 = \chi_4 + \chi_5$.

Exercise 3.1. *Show that the character of the left regular representation of a finite group G is equal to $|G|\delta_e$. Show that each irreducible representation representation of G occurs in the left regular representation with multiplicity equal to the dimension of that irreducible representation. Conclude that the sum of the squares of the dimensions of the irreducible representations is equal to the order of the group.*

Exercise 3.2. *Let $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathrm{SL}_2(\mathbb{C})$ with*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Check that the relations

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$$

hold and conclude that Q is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, the so called quaternion group. *Show that $Q/\{\pm 1\} \cong \mathrm{V}_4$. Determine the conjugation classes in Q. Find the character table of Q.*

Exercise 3.3. *Show that each irreducible character of a finite direct product group $G_1 \times G_2$ is of the form $G_1 \times G_2 \ni (x,y) \mapsto \chi_1(x)\chi_2(y)$ with χ_1 an irreducible character of G_1 and χ_2 an irreducible character of G_2.*

Exercise 3.4. *Show that the irreducible characters of the group $\mathfrak{A}_5 \times \{\pm 1\}$ of reflections and rotations of the icosahedron are of the form*

$$\chi_\pm(x,1) = \chi(x), \chi_\pm(x,-1) = \pm\chi(x)$$

for $x \in \mathfrak{A}_5$ and χ an irreducible character of $\mathfrak{A}_5$.

Exercise 3.5. *Show that a representation (ρ, V) of a finite (or even compact) group G is irreducible if its character $\chi \in \mathrm{C}(G)$ has norm 1 for the standard Hermitian inner product on $\mathrm{L}(G)$.*

4 Molecular vibrations after Wigner

Consider a molecule M in $\mathbb{R}^3$ with n atoms numbered $1, \cdots, n$. Suppose that $q = (q_1, \cdots, q_n) \in \mathbb{R}^{3n}$ is an equilibrium position of M with $q_i \in \mathbb{R}^3$ the position of the atom with number i. We consider small vibrations $x = (x_1, \cdots, x_n) \in \mathbb{R}^{3n}$ in time t from the stationary equilibrium position q, and so M is in position $q + x$. The kinetic energy K of this vibration $x = x(t)$ is equal to

$$K(\dot{x}) = (\dot{x}|\dot{x})/2$$

with $(x|y) = ((x_1, \cdots, x_n)|(y_1, \cdots, y_n)) = \sum_i m_i(x_i, y_i)$ with m_i the mass of the atom with number i and $(\cdot, \cdot)$ the standard inner product on $\mathbb{R}^3$. The potential energy V is in harmonic approximation a homogeneous quadratic polynomial

$$V(x) = (x|Hx)/2$$

with $H : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ a symmetric operator relative to $(\cdot|\cdot)$, that is $(Hx|y) = (x|Hy)$ for all $x, y \in \mathbb{R}^{3n}$. Indeed, we may assume $V(0) = 0$ because the potential is determined from the conservative force field $F : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ by the equation $F(x) = -\nabla V(x)$. The linear part of V vanishes because $x = 0$ is an equilibrium position and so $F(0) = 0$. Cubic and higher order terms of V are ignored because x is small. Equivalently, the force field $F(x) = -Hx$ is a linear vector field. The equation of motion in the *vibration space* $\mathbb{R}^{3n}$ becomes

$$\ddot{x} + Hx = 0$$

by Newton's law.

Define the translation subspace T and the rotation subspace R of $\mathbb{R}^{3n}$ by

$$\begin{aligned} T &= \{(u, \cdots, u) \in \mathbb{R}^{3n}; u \in \mathbb{R}^3\} \\ R &= \{(v \times q_1, \cdots, v \times q_n); v \in \mathbb{R}^3\} \end{aligned}$$

with $\times$ the vector product on $\mathbb{R}^3$. If we assume that the potential energy V is invariant under translations and rotations of M as a a whole, that is there are no external forces on M, then $V(x) = 0$ for $x \in T + R$. If we assume that M is not collinear then it can be shown that $\dim R = 3$ and $T \cap R = \{0\}$, and so we have an orthogonal direct sum decomposition

$$\mathbb{R}^{3n} = V \oplus T \oplus R$$

with V the so called *internal vibration space* of dimension $3n-6$. Since $H=0$ on $T+R$ and H is symmetric we have $H(V)\subset V$.

Finally we assume that q is a stable equilibrium position of M. This means that all eigenvalues of H on V are strictly positive, and the square roots of these eigenvalues are the *frequencies* of the eigenvibrations, after reduction of translation and rotation symmetry. In general, if all eigenvalues of H on V have multiplicity one, then one measures $(3n-6)$ different frequencies. However eigenspaces of H on V might have dimension ≥ 2, in which case one speaks of (spectral) *degeneration*. The principal cause for degeneration is *symmetry* of M as will be explained below.

Suppose that the atoms with number i and number j are of the same kind if an only if $m_i=m_j$. If the center of gravity $\sum m_iq_i/\sum m_i$ of M is taken at the origin of $\mathbb{R}^3$ then the symmetry group G of the rigid body M at rest in equilibrium position $q=(q_1,\cdots,q_n)\in\mathbb{R}^{3n}$ is given by

$$G=\{a\in \mathrm{O}(\mathbb{R}^3);\forall i\ \exists j \text{ with } m_i=m_j \text{ and } aq_i=q_j\}.$$

As subgroup of $\mathrm{O}(\mathbb{R}^3)$ the group G has a standard representation π on $\mathbb{R}^3$ and we write $\chi(a)=\operatorname{tr} a$ for its character. There is a natural homomorphism $G\to\mathfrak{S}_n, a\mapsto\sigma_a$ given by $\sigma_a(i)=j$ if and only if $aq_i=q_j$. We are now able to define the *vibration representation* Π from G on the vibration space $\mathbb{R}^{3n}$ by

$$\Pi(a)(x_1,\cdots,x_n)=(ax_{\sigma_a^{-1}(1)},\cdots,ax_{\sigma_a^{-1}(n)})$$

for $a\in G$ and $x\in\mathbb{R}^{3n}$. Indeed, one thinks of $x=(x_1,\cdots,x_n)\in\mathbb{R}^{3n}$ as a set of arrows $x_j\in\mathbb{R}^3$ with begin point q_j and for $a\in G$ the new arrow from $\Pi(a)x\in\mathbb{R}^{3n}$ with begin point $q_j=aq_i$ (and so $\sigma_a(i)=j$) is just $ax_i\in\mathbb{R}^3$. The next theorem is called *Wigner's rule*.

Theorem 4.1. *The character X of the vibration representation Π of the symmetry group G of the molecule M on the total vibration space $\mathbb{R}^{3n}$ is given by*

$$X(a)=|\{i;\sigma_a(i)=i\}|\cdot\chi(a)$$

for $a\in G$.

Proof. Think of the $3n\times 3n$ matrix $\Pi(a)$ with scalar entries as a $n\times n$ matrix with entries from $\mathrm{End}(\mathbb{R}^3)$. The matrix $\Pi(a)$ is then the permutation matrix of σ_a with nonzero matrix entries a. Hence the matrix $\Pi(a)$ has on the main diagonal place (i,i) the entry a from $G<\mathrm{O}(\mathbb{R}^3)$ if and only if $\sigma_a(i)=i$ and 0 from $\mathrm{End}(\mathbb{R}^3)$ otherwise. Therefore Wigner's rule is obvious. $\square$

It is easy to check that the direct sum decomposition $\mathbb{R}^{3n} = V \oplus T \oplus R$ is invariant for the representation Π of G, and in fact we have

$$\Pi = \Pi_V \oplus \Pi_T \oplus \Pi_R \sim \Pi_V \oplus \pi \oplus (\det \otimes \pi).$$

Indeed $\Pi_T \sim \pi$ is obvious, while $\Pi_R \sim \det \otimes \pi$ follows from $a(u \times v) = \det(a)(au \times av)$ for all $a \in \mathrm{O}(\mathbb{R}^3)$ and $u, v \in \mathbb{R}^3$. Hence the character

$$X_V = X - \chi - \det \cdot \chi$$

of the internal vibration representation Π_V can be computed using Wigner's rule.

Since the potential energy V is invariant under G, that is since we have $V(\Pi(a)x) = V(x)$ for all $a \in G$ and $x \in \mathbb{R}^{3n}$, we get the commutation relation

$$\Pi(a)H = H\Pi(a)$$

for all $a \in G$. In other words H is an intertwiner for Π. This means that the eigenvalue decomposition

$$V = \oplus_\nu V_\nu = \oplus_\nu \{v \in V; Hv = \nu^2 v\}$$

is invariant under Π. If the subrepresentation Π_ν on V_ν is irreducible for all $\nu > 0$ then we say that the operator H_V on V has *natural degeneration* for G. If V_ν is reducible for some $\nu > 0$ then we speak of *accidental degeneration*. Accidental degeneration might hint at a larger *hidden* symmetry group.

Let us discuss the example of the methane molecule CH_4 with a carbon atom at the origin and four hydrogen atoms at the four vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ of a tetrahedron. The symmetry group G is the reflection and rotation group of the tetrahedron and is isomorphic to $\mathfrak{S}_4$.

From the character table of $\mathfrak{S}_4$ one gets the table

$\mathfrak{S}_4$	e	(12)	(12)(34)	(123)	(1234)
$\lvert \cdot \rvert$	1	6	3	8	6
χ_1	1	1	1	1	1
$\det = \chi_2$	1	-1	1	1	-1
χ_3	2	0	2	-1	0
$\chi = \chi_4$	3	1	-1	0	-1
χ_5	3	-1	-1	0	1
X	15	3	-1	0	-1
X_V	9	3	1	0	-1

with $|\cdot|$ the cardinality of the conjugation class, which in turn implies

$$\begin{aligned}
\langle X_V, \chi_1 \rangle &= (1 \cdot 9 \cdot 1 + 6 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot 1 + 8 \cdot 0 \cdot 1 + 6 \cdot -1 \cdot 1)/24 = 1 \\
\langle X_V, \chi_2 \rangle &= (1 \cdot 9 \cdot 1 + 6 \cdot 3 \cdot -1 + 3 \cdot 1 \cdot 1 + 8 \cdot 0 \cdot 1 + 6 \cdot -1 \cdot -1)/24 = 0 \\
\langle X_V, \chi_3 \rangle &= (1 \cdot 9 \cdot 2 + 6 \cdot 3 \cdot 0 + 3 \cdot 1 \cdot 2 + 8 \cdot 0 \cdot -1 + 6 \cdot -1 \cdot 0)/24 = 1 \\
\langle X_V, \chi_4 \rangle &= (1 \cdot 9 \cdot 3 + 6 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot -1 + 8 \cdot 0 \cdot 0 + 6 \cdot -1 \cdot -1)/24 = 2 \\
\langle X_V, \chi_5 \rangle &= (1 \cdot 9 \cdot 3 + 6 \cdot 3 \cdot -1 + 3 \cdot 1 \cdot -1 + 8 \cdot 0 \cdot 0 + 6 \cdot -1 \cdot 1)/24 = 0
\end{aligned}$$

and hence

$$X_V = \chi_1 + \chi_3 + 2\chi_4.$$

The conclusion is that the internal vibration spectrum of methane in case of natural degeneration has 4 frequencies, one mode transforming under ρ_1 and with multiplicity $\dim \rho_1 = 1$, one mode transforming under ρ_3 with multiplicity $\dim \rho_3 = 2$ and two modes transforming under ρ_4 with multiplicity $\dim \rho_4 = 3$.

The symmetry results of this section give only qualitative information about the nature of spectral degeneration. For finer quantitative information about the location of the spectral lines one needs the further knowledge about the masses m_i and the Hessian H of the potential.

Remark 4.2. *Usually in representation theory one considers representations on* complex *vector spaces. However, in this section we have tacitly worked with representations on* real *vector spaces. So some care is required with the concept of irreducible characters over $\mathbb{R}$ or $\mathbb{C}$, in the sense that a real irreducible character can be either a complex irreducible character or a sum of a complex irreducible character and its complex conjugate. However, in the two examples discussed in this section, the group of reflections and rotations of the tetrahedron (in the case of methane) or the icosahedron (in the case of the buckyball) all complex irreducible characters are real valued, and real irreducible characters are complex irreducible characters as well.*

Symmetry arguments also give so called *selection rules.* If one shines light on the molecule M with vibrations of M as a result then only light of particular wave length is absorbed in accordance with the particular frequencies $\nu > 0$ of M (or equivalently with particular eigenvalues ν^2 of the symmetric operator H on V). If the spectrum has natural degeneration, then to each frequency ν one associates the irreducible representation ρ_ν of G on the eigenspace V_ν of H on V.

In ordinary spectroscopy one will only see those frequencies ν of M for which the irreducible representation ρ_ν occurs as a subrepresentation of the standard representation ρ of G on $\mathbb{R}^3$. This ordinary spectrum is usually seen in infrared. There is also the so called Raman spectrum, which sees only those frequencies ν for which the irreducible representation ρ_ν occurs as subrepresentation of the second symmetric power $S^2\rho$ of the standard representation ρ of G on $\mathbb{R}^3$. The Raman spectrum is a second order scattering effect, and is usually seen in ultraviolet. For the mathematics behind these selection rules see the text book Shlomo Sternberg, Group theory and physics, Cambridge University Press, 1994.

The results of this section were obtained by Eugene Wigner (1902-1995) in 1930. In the nineteen twenties and thirties Wigner made various significant applications of representation theory of groups to physics, for which he was awarded the Nobel prize in physics in 1963.

Exercise 4.1. *Show that the dimension of the rotation subspace R of $\mathbb{R}^{3n}$ is equal to 3 if $\mathrm{rk}\{q_1, \cdots, q_n\} \geq 2$. Show that $T \cap R = \{0\}$ if $q_1, \cdots, q_n$ are not collinear.*

Exercise 4.2. *Show that for a molecule M whose symmetry group G contains the central inversion -1 the ordinary spectrum and the Raman spectrum are disjoint (under the usual assumption of natural degeneration).*

Exercise 4.3. *For $G < \mathrm{O}_3(\mathbb{R})$ a finite group let π denote the standard three dimensional representation and χ its character. For $u \in \mathbb{R}^3$ let $L(u)$ be the skew symmetric linear operator on $\mathbb{R}^3$ defined by $L(u)v = u \times v$. Show that L is an intertwiner from $\det \otimes \pi$ to $\wedge^2\pi$ and conclude that the character of $S^2\pi$ is equal to $(\chi - \det)\chi$.*

Exercise 4.4. *The* buckyball C_{60} *is a molecule with 60 carbon atoms at the vertices of a truncated icosahedron. The symmetry group of the buckyball is the reflection and rotation group of the icosahedron and is isomorphic to $\mathfrak{A}_5 \times \{\pm 1\}$. Show that the character X of the natural representation Π of the symmetry group $G \cong \mathfrak{A}_5 \times \{\pm 1\}$ of the buckyball on the total vibration space $\mathbb{R}^{180}$ takes the value 180 at the identity element e, the value 4 at the class of the reflection $-a$ and the value 0 otherwise. If $\psi = \sum \chi_{i,+} + \sum \chi_{i,-}$ in the notation of Exercise 3.4 then check that $\psi(e) = 32$ and $\psi(-a) = 0$. Show that $\langle X, \psi \rangle = 48$ and conclude that the internal vibration spectrum of C_{60} in case of natural degeneration has 46 frequencies. Show that the ordinary* *spectrum of* C_{60} *has only* 4 *frequencies, while the Raman spectrum has* 10 *frequencies.*

The German-Canadian physicist and physical chemist Gerhard Herzberg (1904-1999), who won the Nobel prize for Chemistry in 1971, wrote in the nineteen fifties a highly influential four volume text book Molecular Spectra and Molecular Structure, sometimes referred to as the bible of spectroscopy. In volume I he lists all the finite subgroups of the orthogonal group $O_3(\mathbb{R})$ of Euclidean space as possible molecular symmetry groups. The icosahedral group occurred in his list, but he added in a footnote that it is highly unlikely that this group will appear in nature as symmetry group. The experimental discovery in 1985 of C_{60} with its rich icosahedral symmetry came therefore as a big surprise. For this work Curl, Kroto and Smalley were awarded the Nobel prize for Chemistry in 1996. For their discovery of the flat analogue of the bucky ball, the so called graphene molecule, Andre Geim and Konstantin Novoselov were awarded the Nobel prize for Physics in 2010. Earlier that year Andre Geim was appointed honorary professor at the Radboud University Nijmegen, helping our university to win its first Nobel prize.

5 The spin homomorphism

Since for $z_1, z_2, w_1, w_2 \in \mathbb{C}$

$$\begin{pmatrix} z_1 & w_1 \\ -\overline{w}_1 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} z_2 & w_2 \\ -\overline{w}_2 & \overline{z}_2 \end{pmatrix} = \begin{pmatrix} z_1 z_2 - w_1 \overline{w}_2 & z_1 w_2 + w_1 \overline{z}_2 \\ -\overline{w}_1 z_2 - \overline{z}_1 \overline{w}_2 & -\overline{w}_1 w_2 + \overline{z}_1 \overline{z}_2 \end{pmatrix}$$

it follows that

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} ; z, w \in \mathbb{C} \right\}$$

is an associative algebra over $\mathbb{R}$ of dimension 4, called the *quaternion* algebra. They were introduced by William Hamilton (1805-1865) on Monday 16 October 1843, who carved their multiplication rule

$$i^2 = j^2 = k^2 = ijk = -1$$

for the real basis

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

into a stone of Brougham Bridge over the Royal Canal in Dublin, as he paused on it. We shall write quaternions as

$$q = z + wj = u_0 + u_1 i + u_2 j + u_3 k$$

with $z = u_0 + u_1 i, w = u_2 + u_3 i \in \mathbb{C}$ and $u_0, u_1, u_2, u_3 \in \mathbb{R}$. We denote by $\overline{q} = u_0 - u_1 i - u_2 j - u_3 k$ the *conjugate* quaternion of q. The *norm* $|q| = \sqrt{\det q} = \sqrt{q\overline{q}}$ is a multiplicative map from $\mathbb{H}$ to $\mathbb{R}_{\geq 0}$, turning $\mathbb{H}$ into an associative *division algebra*. This means that every nonzero quaternion q has an inverse, indeed namely $\overline{q}/|q|^2$. We shall denote $\Re q = u_0 \in \mathbb{R}$ for the *real part* and $\Im q = u_1 i + u_2 j + u_3 k \in \mathbb{R}^3$ for the *imaginary part* of $q \in \mathbb{H}$. A quaternion q is called real if $q = \Re q$ and purely imaginary if $q = \Im q$.

The norm 1 quaternions form a group denoted $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{U}_1(\mathbb{H})$ just like the norm 1 complex numbers form a group denoted $\mathrm{SO}_2(\mathbb{R}) \cong \mathrm{U}_1(\mathbb{C})$. Both $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SO}_2(\mathbb{R})$ are smooth manifolds, the latter the unit circle and the former the unit sphere of dimension 3. The multiplication and inversion on $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SO}_2(\mathbb{R})$ is given by smooth maps and as such these are primary examples of compact Lie groups.

Let $\mathrm{SO}_3(\mathbb{R})$ be the special orthogonal group of the Euclidean space $\mathbb{R}^3$. In the sequel we shall identify $\mathbb{R}^3$ with the subspace of purely imaginary quaternions in $\mathbb{H}$. The map

$$\pi : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{R}), \pi(q)(u) = qu\overline{q}$$

for $q \in \mathrm{SU}_2(\mathbb{C})$ and $u \in \mathbb{R}^3$ is called the *spin homomorphism.*

Theorem 5.1. *The spin homomorphism $\pi : \mathrm{SU}_2(\mathbb{C}) \twoheadrightarrow \mathrm{SO}_3(\mathbb{R})$ has image $\mathrm{SO}_3(\mathbb{R})$ and kernel ± 1.*

Proof. For $q \in \mathbb{H}$ we have $q \in \mathrm{SU}_2(\mathbb{C})$ if an only if $\overline{q} = q^{-1}$ which in turn implies that $\pi : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{R})$ is a homomorphism. In order to see that the image of π is contained in $\mathrm{SO}_3(\mathbb{R})$ we use the formula

$$uv = (u, v) + u \times v$$

for $u, v \in \mathbb{R}^3$ expressing the multiplication of purely imaginary quaternions in terms of the scalar product $(\cdot, \cdot)$ and vector product $\cdot \times \cdot$ on $\mathbb{R}^3$. It follows that

$$(\pi(q)u, \pi(q)v) = (u, v), (\pi(q)u) \times (\pi(q)v) = \pi(q)(u \times v)$$

for all $u, v \in \mathbb{R}^3$ and therefore $\pi(q) \in \mathrm{SO}_3(\mathbb{R})$ for all $q \in \mathrm{SU}_2(\mathbb{C})$.

Since the center of $\mathbb{H}$ is equal to $\mathbb{R}$ it follows that $q \in \mathrm{SU}_2(\mathbb{C})$ commutes with all purely imaginary quaternions if and only if $q = \pm 1$.

It remains to check that the image of π equals $\mathrm{SO}_3(\mathbb{R})$. It is known (and due to Leonard Euler (1707-1783) in his work on rigid body motions) that each element r of $\mathrm{SO}_3(\mathbb{R})$ is a rotation around a directed axis $\mathbb{R}w$ for some unit vector $w \in \mathbb{R}^3$ over an angle $\theta \in \theta \in \mathbb{R}/2\pi\mathbb{Z}$. In other words, such a rotation r is given by

$$ru = u \cos\theta + v \sin\theta, rv = -u \sin\theta + v \cos\theta, rw = w$$

in an orthonormal basis $\{u, v, w\}$ of $\mathbb{R}^3$ with $u \times v = w$. The straightforward verification that the unit quaternion $q = \cos\theta + w \sin\theta$ satisfies $\pi(q) = r$ is left to the reader. Hence the theorem follows. □

By the homomorphism theorem it follows that $\mathrm{SU}_2(\mathbb{C})/\{\pm 1\} \cong \mathrm{SO}_3(\mathbb{R})$. Geometrically the group $\mathrm{SU}_2(\mathbb{C})$ is the unit sphere S^3 of dimension 3. The spin homomorphism identifies antipodal points and so geometrically $\mathrm{SO}_3(\mathbb{R})$

is just the real projective space $\mathbb{P}^3(\mathbb{R})$ of dimension 3. The orthogonal projection map

$$\Re : \mathrm{SU}_2(\mathbb{C}) \to [-1, 1]$$

is a class function on $\mathrm{SU}_2(\mathbb{C})$ and the conjugation classes are in fact the level surfaces of this map. The two conjugation classes $\Re^{-1}(\pm 1) = \pm 1$ are just points and the remaining classes are spheres of dimension 2. Hence the circle subgroup $\mathbb{R}/2\pi\mathbb{Z} \cong \mathrm{U}_1(\mathbb{C})$ of $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{U}_1(\mathbb{H})$ intersects each conjugation class in at most 2 points, which are inverses of each other.

Lemma 5.2. *If μ_E is the Euclidean measure on the unit sphere S^3 in $\mathbb{H}$ then for each class function $\phi \in \mathrm{C}(\mathrm{SU}_2(\mathbb{C}))$ we have the integral formula*

$$\int_{S^3} \phi(x) d\mu_E(x) = 2\pi \int_0^{2\pi} \phi(\cos\theta + i \sin\theta) \sin^2\theta d\theta$$

and so the Euclidean volume of S^3 is equal to $2\pi^2$.

Proof. For $0 < \theta < \pi$ the conjugation class of the element $(\cos\theta + i \sin\theta)$ in $\mathrm{SU}_2(\mathbb{C})$ is a sphere of dimension 2 with radius $\sin\theta$ and so with Euclidean area $4\pi \sin^2\theta$. Since class functions on $\mathrm{SU}_2(\mathbb{C})$ restrict to even even functions on $\mathrm{U}_1(\mathbb{C}) \cong \mathbb{R}/2\pi\mathbb{Z}$ the integral formula follows from calculus. The Euclidean volume of S^3 follows from this integral formula by taking $\phi = 1$. □

Corollary 5.3. *If μ is the normalized invariant measure on $\mathrm{SU}_2(\mathbb{C})$ then*

$$\int_{\mathrm{SU}_2(\mathbb{C})} \phi(x) d\mu(x) = \frac{1}{4\pi} \int_0^{2\pi} \phi(\cos\theta + i \sin\theta) \Delta(\theta) \overline{\Delta(\theta)} d\theta$$

with $\phi \in \mathrm{C}(\mathrm{SU}_2(\mathbb{C}))$ a class function on $\mathrm{SU}_2(\mathbb{C})$ and $\Delta(\theta) = (e^{i\theta} - e^{-i\theta})$ an odd integral Fourier polynomial on $\mathrm{U}_1(\mathbb{C}) \cong \mathbb{R}/2\pi\mathbb{Z}$.

Theorem 5.4. *The restriction of the irreducible characters of $\mathrm{SU}_2(\mathbb{C})$ to $\mathrm{U}_1(\mathbb{C})$ are of the form*

$$\chi_n(\cos\theta + i \sin\theta) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(n+1)\theta}{\sin\theta}$$

for some $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$.

Proof. Let (ρ, V) be an irreducible representation of $\mathrm{SU}_2(\mathbb{C})$ with irreducible character $\chi = \operatorname{tr} \rho$ a class function on $\mathrm{SU}_2(\mathbb{C})$. The irreducible representations of the circle group $\mathrm{U}_1(\mathbb{C}) \cong \mathbb{R}/2\pi\mathbb{Z}$ were shown to be of the form $\rho_n(e^{i\theta}) = e^{in\theta}$ for some $n \in \mathbb{Z}$. The restriction $\chi(\cos\theta + i\sin\theta)$ of χ to $\mathrm{U}_1(\mathbb{C})$ is an even integral Fourier polynomial, and so the function $\chi(\cos\theta + i\sin\theta)\Delta(\theta)$ is an odd integral Fourier polynomial. Note that the Fourier coefficients of the $\chi(\cos\theta + i\sin\theta)$ are all ≥ 0.

Taking $\phi = \chi\overline{\chi}$ in the integral formula of the above corollary yields

$$\frac{1}{2\pi}\int_0^{2\pi} \chi(\cos\theta + i\sin\theta)\Delta(\theta)\overline{\chi(\cos\theta + i\sin\theta)\Delta(\theta)}d\theta = 2$$

by the Schur orthogonality relation $\langle \chi, \chi \rangle = 1$. Writing

$$\chi(\cos\theta + i\sin\theta)\Delta(\theta) = \sum a_n e^{in\theta}$$

this amounts to $\sum a_n \overline{a}_n = 2$ and since $a_{-n} = -a_n$ also to $\sum_{n>0} a_n \overline{a}_n = 1$. By integrality of a_n we conclude that all a_n for $n > 0$ are equal to 0 with the exception of one, which is equal to ± 1. In other words, we have

$$\chi_n(\cos\theta + i\sin\theta)\Delta(\theta) = \pm(e^{i(n+1)\theta} - e^{-i(n+1)\theta})$$

for some $n \geq 0$. Since all Fourier coefficients of $\chi_n(\cos\theta + i\sin\theta)$ are ≥ 0 the $\pm$ becomes $+$ and the theorem follows. $\square$

Remark 5.5. *The irreducible representation (ρ_n, V_n) of $\mathrm{SU}_2(\mathbb{C})$ with the irreducible character χ_n as in the theorem has dimension $\chi_n(1) = (n+1)$ by l' Hopital's rule. If we denote by ρ the standard representation of $\mathrm{SU}_2(\mathbb{C})$ on $V = \mathbb{C}^2$ then it is easy to check that the representation $\rho_n = S^n\rho$ on the space $V_n = S^nV$ of binary forms of degree n has character equal to*

$$\chi_n(\cos\theta + i\sin\theta) = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-i(n-2)\theta} + e^{-in\theta}$$

and coincides with the character found in the above theorem. The conclusion is that the degree n binary forms representation of $\mathrm{SU}_2(\mathbb{C})$ is irreducible and in fact these are all the irreducible representations of $\mathrm{SU}_2(\mathbb{C})$ up to equivalence.

Remark 5.6. *Mathematicians parametrize the irreducible representations of $\mathrm{SU}_2(\mathbb{C})$ by the* degree *$n \in \mathbb{N}$ of the binary forms. However, physicists* *sometimes use the* spin $l = \frac{1}{2}n \in \frac{1}{2}\mathbb{N}$ *instead. Since* $\rho_n(-1) = (-1)^n$ *the irreducible representation* ρ_n *descends to the rotation group* $\mathrm{SO}_3(\mathbb{R})$ *in case the degree* n *is even or equivalently the spin* $l = \frac{1}{2}n$ *is integral. The irreducible representations with half integral spin are no longer honest representations of the rotation group* $\mathrm{SO}_3(\mathbb{R})$ *but only of the double spin cover* $\mathrm{SU}_2(\mathbb{C})$*.*

The group $\mathrm{SU}_2(\mathbb{C})$ is the simplest (nontrivial) example of a connected simply connected compact Lie group. The method of this section has been generalized by Hermann Weyl (1885-1955) in 1925 to any connected simply connected compact Lie group. Examples of such Lie groups are the special unitary group $\mathrm{SU}_n(\mathbb{C})$ for $n \geq 2$ and the spin group $\mathrm{Spin}_n(\mathrm{R})$ for $n \geq 3$. The special orthogonal group $\mathrm{SO}_n(\mathbb{R})$ for $n \geq 3$ is a connected compact Lie group with fundamental group of order 2. Its universal (double) cover is a connected simply connected compact Lie group, which is denoted $\mathrm{Spin}_n(\mathbb{R})$ and called the spin cover of $\mathrm{SO}_n(\mathbb{R})$. In this section we have shown $\mathrm{Spin}_3(\mathbb{R}) = \mathrm{SU}_2(\mathbb{C})$ and from Exercise 5.3 it follows that $\mathrm{Spin}_4(\mathbb{R}) = \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$.

Now let G be a connected simply connected compact Lie group. One chooses a maximal torus $T \cong \mathbb{R}^n/2\pi L$ in G with L a lattice in $\mathbb{R}^n$. It can be shown that any two maximal tori in G are conjugated, which in turn implies that the dimension n of T is an invariant of G, called the rank of G. The rank of $\mathrm{SU}_2(\mathbb{C})$ is one and in fact $\mathrm{SU}_2(\mathbb{C})$ is the only connected simply connected compact Lie group of rank one.

Any element of G is contained in some maximal torus of G, and hence the character χ of an irreducible representation of G is completely determined by its restriction $\chi|_T$ to the torus T. By representation theory of T and structure theory of G it follows that $\chi|_T$ is an integral Fourier polynomial on T, which is invariant under the Weyl group $W = N/T$ with $N = \{x \in G; xtx^{-1} \in T \ \forall t \in T\}$ the normalizer of T in G. The Weyl group W acts on T by conjugation as a finite group generated by reflections. Weyl group invariant Fourier polynomials on T are just even Fourier polynomials on the the circle group $\mathbb{R}/2\pi\mathbb{Z}$ in the $\mathrm{SU}_2(\mathbb{C})$ case.

The irreducibility criterion $\langle \chi, \chi \rangle = 1$ can be used along the same lines as above for $\mathrm{SU}_2(\mathbb{C})$ to write $\chi|_T$ as the quotient of two alternating integral Fourier polynomials on T. This is the famous Weyl character formula, and is considered one of the highlights of Weyl's work. Weyl's approach is transcendental using integration theory. A purely algebraic approach to Weyl's character formula was found by the German-Dutch mathematician Hans Freudenthal (1905-1990) in 1954.

The original approach of Weyl is explained for example in the text book J.J. Duistermaat and J.A.C. Kolk, Lie groups, Springer, 1999. The text book H. Freudenthal and H. de Vries, Linear Lie Groups, Academic Press, 1969 was the first modern exposition of the subject, and explains both approaches. It has interesting historical comments. Unfortunately, the notation is quite unusual, probably due to certain ideas of Freudenthal on the didactics of mathematics. This is maybe the main reason that the book did not have the influence, which it deserved. Henk de Vries was my former colleague here in Nijmegen. Hans Duistermaat, one of my two PhD advisors, was PhD student and successor of Hans Freudenthal in Utrecht. The mathematical institute in Utrecht is located in the Freudenthal building, named in 2013 after Freudenthal, to commemorate his contributions for Dutch mathematics.

Exercise 5.1. *Show Euler's result that each element $r \in \mathrm{SO}_3(\mathbb{R})$ is a rotation around some axis over some angle, and so of the form*

$$ru = u\cos\theta + v\sin\theta, rv = -u\sin\theta + v\cos\theta, rw = w$$

for some orthonormal basis $\{u, v, w\}$ of $\mathbb{R}^3$ with $u \times v = w$.

Exercise 5.2. *Explain in the setting of the previous section why the rotation subspace R of $\mathbb{R}^{3n}$ was defined by $R = \{(v \times q_1, \cdots, v \times q_n); v \in \mathbb{R}^3\}$.*

Exercise 5.3. *The group $\mathrm{SO}_4(\mathbb{R})$ is the connected group of isometries of the round sphere $S^3 \cong \mathrm{SU}_2(\mathbb{C})$ in $\mathbb{R}^4 \cong \mathbb{H}$. Show that there exists a spin double cover homomorphism*

$$\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \twoheadrightarrow \mathrm{SO}_4(\mathbb{R})$$

with kernel $\{\pm(-1, -1)\}$ of order 2. Conclude that the equivalence classes of irreducible representations of $\mathrm{SO}_4(\mathbb{R})$ are parametrized by pairs $(m, n) \in \mathbb{N}^2$ of nonnegative integers with $m + n$ even.

Exercise 5.4. *Show using character theory the Clebsch-Gordan rule*

$$\rho_n \otimes \rho_m \sim \rho_{n+m} \oplus \rho_{n+m-2} \oplus \cdots \oplus \rho_{|n-m|+2} \oplus \rho_{|n-m|}$$

for the decomposition of the tensor product of two irreducible representations of $\mathrm{SU}_2(\mathbb{C})$ as direct sum of irreducible representations.

6 Lie groups and Lie algebras

Let M be a smooth manifold. The linear space $C^\infty(M)$ of smooth real valued functions on M is a commutative algebra with respect to pointwise multiplication of functions. Let us denote by T_xM the tangent space of M at the point $x \in M$. A smooth vector field X on M assigns to each point $x \in M$ a tangent vector $X_x \in T_xM$ such that X_x varies smoothly with $x \in M$. The vector space of smooth vector fields on M is denoted $\mathfrak{X}(M)$, and is in fact a left module over $C^\infty(M)$.

Given a smooth vector field X on M there exists for each $x \in M$ a unique smooth solution curve

$$\gamma = \gamma_x : I_x \to M, \dot{\gamma}(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = X_{\gamma(t)}, \gamma(0) = x$$

defined on a maximal open interval I_x around 0 in $\mathbb{R}$. In local coordinates this is just the existence and uniqueness theorem for a first order system of ordinary differential equations. If $s \in I_x$ and $t \in I_{\gamma_x(s)}$ then it is clear that

$$\gamma_x(t+s) = \gamma_{\gamma_x(s)}(t)$$

and so $(t+s) \in I_x$. Let us also write $\phi_t(x) = \gamma_x(t)$. The map $\phi_t : D_t \to M$ is smooth with domain

$$D_t = \{x \in M; t \in I_x\}$$

and is called the flow of X after time t. The flow satisfies the group property

$$\phi_{t+s}(x) = \phi_t(\phi_s(x))$$

for $x \in D_s$ and $\phi_s(x) \in D_t$. The flows $\{\phi_t\}_{t\in\mathbb{R}}$ are called a local *one parameter group of diffeomorphisms.* Only in case $X \in \mathfrak{X}(M)$ is complete, which means that $D_t = M$ for all $t \in \mathbb{R}$, one gets a true one parameter subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ of M.

If $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ then the *directional derivative* or *Lie derivative* of f in the direction of X

$$\mathcal{L}_X(f)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\phi_t^* f(x) = f(\phi_t(x))\Big\}_{t=0}$$

is again a smooth function on M. The derivative is a *derivation* in the sense that

$$\mathcal{L}_X(fg) = \mathcal{L}_X(f)g + f\mathcal{L}_X(g)$$

for all $X \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Moreover it is *local* in the sense that the value of $\mathcal{L}_X(f)$ at some point $x \in M$ only depends on the restrictions X_U and f_U of X and f to an open neighborhood U of x in M. Let us denote by $\mathrm{Der}(C^\infty(M))$ the vector space of local derivations of $C^\infty(M)$. The linear map

$$\mathfrak{X}(M) \to \mathrm{Der}(C^\infty(M)), X \mapsto \mathcal{L}_X$$

is in fact a linear isomorphism.

The Lie derivative $\mathcal{L}_X$ extends $C^\infty(M) = \Omega^0(M)$ to the space $\Omega^p(M)$ of differential forms of degree p by the same formula

$$\mathcal{L}_X(\alpha)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\phi_t^*\alpha)\Big\}_{t=0}$$

as in the $\alpha = f \in C^\infty(M)$. We have

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$$

for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, and so $\mathcal{L}_X$ is a derivation of the algebra $(\Omega(M), \wedge)$. The Lie derivative $\mathcal{L}_X$ is also defined on $\mathfrak{X}(M)$ by the formula

$$\mathcal{L}_X(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\phi_t^* Y\Big\}_{t=0}$$

for $Y \in \mathfrak{X}(M)$ and ϕ_t the flow of X at time t. We also write $\mathcal{L}_X(Y) = [X, Y]$ and call it the Lie bracket of $X, Y \in \mathfrak{X}(M)$. From this definition it is clear that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$. This relation is called the Jacobi identity, and can be written in the equivalent form $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ with the first bracket $[\cdot, \cdot]$ the commutator bracket in $\mathrm{End}(C^\infty(M))$ of $\mathrm{End}(\Omega^p(M))$ or $\mathrm{End}(\mathfrak{X}(M))$.

All three notions of $X \in \mathfrak{X}(M)$, namely either as a family of tangent vectors $X_x \in T_xM$ smoothly varying with $x \in M$, or as a local one parameter group $\{\phi_t\}_{t\in\mathbb{R}}$ of diffeomorphisms, or as a derivation $\mathcal{L}_X$ of $C^\infty(M)$ are equivalent ways of thinking about smooth vector fields on M.

Definition 6.1. *A* Lie group *G is both a smooth manifold and a group, and the two structures are compatible in the sense that multiplication and inversion*

$$G \times G \to G, (x, y) \mapsto xy \qquad G \to G, x \mapsto x^{-1}$$

are smooth maps.

Basic examples are the circle group $\mathbb{R}/2\pi\mathbb{Z} \cong \mathrm{SO}_2(\mathbb{R}) = \mathrm{U}_1\mathbb{C}$ and its noncommutative analogue $\mathrm{SU}_2(\mathbb{C}) = \mathrm{U}_1(\mathbb{H})$ as double spin covering of the rotation group $\mathrm{SO}_3(\mathbb{R})$, which is just the group of linear transformations of Cartesian space $\mathbb{R}^3$ preserving both scalar and vector product.

Let G be a Lie group. Let us denote by $L_x : G \to G, L_x(y) = xy$ the smooth map of left multiplication by $x \in G$. In fact L_x is a diffeomorphism of G with inverse $L_{x^{-1}}$. A vector field $X \in \mathfrak{X}(G)$ is called left invariant if $T_y(L_x)X_y = X_{xy}$ for all $x, y \in G$. Taking $y = e$ it follows that left invariant vector fields on G are completely determined by their value at the identity element. The linear subspace of $\mathfrak{X}(G)$ of left invariant vector fields on G is called the Lie algebra of G, and is denoted by $\mathfrak{g}$. Since $\mathfrak{g} \cong T_eG$ the dimension of $\mathfrak{g}$ as vector space is the dimension of G as manifold.

The flow $\phi_t : G \to G$ of $X \in \mathfrak{g}$ is easily seen to be complete, and so we get a one parameter subgroup $\mathbb{R} \to G, t \mapsto \exp(tX) = \phi_t(e)$ of G. The Lie algebra $\mathfrak{g}$ of G consists of the infinitesimal generators of all one parameter subgroups of G. For $X \in \mathfrak{g}$ the Lie derivative $\mathcal{L}_X \in \mathrm{End}(C^\infty(M))$ is given by

$$\mathcal{L}_X(f)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big\{f(x\exp(tX))\Big\}_{t=0}$$

and so $\lambda(x)\mathcal{L}_X = \mathcal{L}_X\lambda(x)$ with λ the left regular representation of G on $C^\infty(G)$, given by $(\lambda(x)f)(y) = f(x^{-1}y)$ as before. Conversely, any local derivation of $C^\infty(G)$ which commutes with the left regular representation is of the form $\mathcal{L}_X$ for some $X \in \mathfrak{g}$. The conclusion is that for $X, Y \in \mathfrak{g}$ the Lie bracket $[X, Y]$ is again left invariant, so $[X, Y] \in \mathfrak{g}$.

Definition 6.2. *A* Lie algebra $\mathfrak{g}$ *over a field F is a finite dimensional vector space over F together with a bilinear operation*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto [X, Y]$$

which is antisymmetric and satisfies the Jacobi identity, so

$$[Y, X] = -[X, Y],\ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

So the vector space $\mathfrak{g}$ of left invariant vector fields on a Lie group G is a real Lie algebra, called the Lie algebra of G.

Most Lie groups occur as subgroup of the *general linear group* $\mathrm{GL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$ with Lie algebra $\mathfrak{gl}_n(\mathbb{R})$. For that reason Hermann Weyl nicknamed the general linear group also *Her All Embracing Majesty.* Its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ is called the *general linear algebra.* As a set $\mathfrak{gl}_n(\mathbb{R})$ is just the matrix algebra $\mathrm{Mat}_n(\mathbb{R})$ but the latter is an associative algebra with respect to matrix multiplication while the former is a Lie algebra with Lie bracket the commutator bracket $[X,Y]=XY-YX$ for $X,Y\in\mathrm{Mat}_n(\mathbb{R})$.

It is a theorem of Élie Cartan that any closed subgroup G of $\mathrm{GL}_n(\mathbb{R})$ is in fact a Lie subgroup with Lie algebra $\mathfrak{g}=\{X\in\mathfrak{gl}_n(\mathbb{R});\exp(tX)\in G\ \forall\, t\in\mathbb{R}\}$. The symmetry group of volume preserving linear transformations of $\mathbb{R}^n$ is a Lie group with Lie algebra

$$\mathrm{SL}_n(\mathbb{R})=\{x\in\mathrm{GL}_n(\mathbb{R});\det x=1\},\ \mathfrak{sl}_n(\mathbb{R})=\{X\in\mathfrak{gl}_n(\mathbb{R});\mathrm{tr}\,x=0\}$$

called the *special linear group* and *special linear algebra* respectively. The symmetry group of Euclidean space $\mathbb{R}^n$ with standard scalar product $(\cdot,\cdot)$ is a Lie group with Lie algebra

$$\mathrm{O}_n(\mathbb{R})=\{x\in\mathrm{GL}_n(\mathbb{R});x^tx=1\},\ \mathfrak{o}_n(\mathbb{R})=\{X\in\mathfrak{gl}_n(\mathbb{R});X^t+X=0\}$$

called the *orthogonal group* and *orthogonal algebra* respectively. The *special orthogonal group* $\mathrm{SO}_n(\mathbb{R})=\mathrm{SL}_n(\mathbb{R})\cap\mathrm{O}_n(\mathbb{R})$ is an index two subgroup of $\mathrm{O}_n(\mathbb{R})$ with the same Lie algebra $\mathfrak{o}_n(\mathbb{R})$. The symmetry group of Hermitian space $\mathbb{C}^n$ with standard scalar product $\langle\cdot,\cdot\rangle$ is a Lie group with Lie algebra

$$\mathrm{U}_n(\mathbb{C})=\{x\in\mathrm{GL}_n(\mathbb{C});x^\dagger x=1\},\ \mathfrak{u}_n(\mathbb{C})=\{X\in\mathfrak{gl}_n(\mathbb{C});X^\dagger+X=0\}$$

called the *unitary group* and *unitary algebra.* The group $\mathrm{SU}_n(\mathbb{C})=\mathrm{SL}_n(\mathbb{C})\cap\mathrm{U}_n(\mathbb{C})$ is called the *special unitary group* and has the *special unitary algebra* $\mathfrak{su}_n(\mathbb{C})=\{X\in\mathfrak{u}_n(\mathbb{C});\mathrm{tr}\,X=0\}$ as its lie algebra. The symmetry group of Hermitian quaternion space $\mathbb{H}^n$ with standard scalar product $\langle\cdot,\cdot\rangle$ is a Lie group with Lie algebra

$$\mathrm{U}_n(\mathbb{H})=\{x\in\mathrm{GL}_n(\mathbb{H});x^\dagger x=1\},\ \mathfrak{u}_n(\mathbb{C})=\{X\in\mathfrak{gl}_n(\mathbb{H});X^\dagger+X=0\}$$

called the *unitary symplectic group* and *unitary symplectic algebra.* The groups $\mathrm{SO}_{n+2}(\mathbb{R}),\mathrm{SU}_{n+1}(\mathbb{C}),\mathrm{U}_n(\mathbb{H})$ for $n\geq 1$ are called the *classical* compact Lie groups. They are all connected and the latter two are simply connected, while $\mathrm{SO}_n(\mathbb{R})$ for $n\geq 3$ has a simply connected double cover group $\mathrm{Spin}_n(\mathbb{R})$. It can be shown that the group $\mathrm{Spin}_n(\mathbb{R})$ is equal to $\mathrm{SU}_2(\mathbb{C}),\mathrm{SU}_2(\mathbb{C})\times\mathrm{SU}_2(\mathbb{C}),\mathrm{U}_2(\mathbb{H}),\mathrm{SU}_4(\mathbb{C})$ for $n=3,4,5,6$ respectively.

If $\pi : G \to H$ is a homomorphism of Lie groups then we get a a derived homomorphism $\pi : \mathfrak{g} \to \mathfrak{h}$ of the associated Lie algebras. By abuse of notation we have used the same letter π for the Lie group and Lie algebra homomorphism. Clearly $\pi : \mathfrak{g} \to \mathfrak{h}$ is a linear map as the tangent map of $\pi : G \to H$ at the identity element, using the identification $\mathfrak{g} = T_eG$. The image under $\pi : G \to H$ of the one parameter subgroup $\exp(tX)$ of G with infinitesimal generator $X \in \mathfrak{g}$ is the one parameter subgroup

$$\pi(\exp(tX)) = \exp(t\pi(X))$$

of H with infinitesimal generator $\pi(X) \in \mathfrak{h}$ for all $t \in \mathbb{R}$. In particular we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\pi} & H \end{array}$$

by taking $t = 1$.

In order that $\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism it remains to understand that $\pi([X, Y]) = [\pi(X), \pi(Y)]$ for all $X, Y \in \mathfrak{g}$. In the case of linear Lie groups $G < \mathrm{GL}_n(\mathbb{R})$ with linear Lie algebra $\mathfrak{g} < \mathfrak{gl}_n(\mathbb{R})$ and like wise for $H < \mathrm{GL}_m(\mathbb{R})$ and $\mathfrak{h} < \mathfrak{gl}_m(\mathbb{R})$ this follows from the formula

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X, Y] + O(t^3))$$

for $t \to 0$ and $X, Y \in \mathfrak{g} < \mathfrak{gl}_n(\mathbb{R})$. Taking $s = t^2$ and neglecting higher order terms shows

$$\frac{d}{ds}\{\pi(\exp(s[X, Y]))\}|_{s=0} = \frac{d}{ds}\{\exp(s[\pi(X), \pi(Y)]))\}|_{s=0}$$

which in turn implies $\pi([X, Y]) = [\pi(X), \pi(Y)]$ for $X, Y \in \mathfrak{g}$.

We have seen that a Lie group homomorphism can be derived to a Lie algebra homomorphism such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\pi} & H \end{array}$$

is commutative. The exponential map is a local diffeomorphism from $(\mathfrak{g}, 0)$ to (G, e) with inverse log. Hence the Lie algebra homomorphism determines the Lie group homomorphism uniquely, as long as G is a connected Lie group. It can be shown that for G connected and simply connected any Lie algebra homomorphism integrates to a unique Lie group homomorphism.

A representation of a Lie group G is just a Lie group homomorphism $G \to \mathrm{GL}(V)$ for some (finite dimensional) complex vector space V. Such a representation can be derived to a Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(V)$ which uniquely determines the original Lie group representation, as long as G is connected. Therefore the both algebraic and analytic problem of finding the representations of a connected Lie group G boils down to two problems. In the first place determine the Lie algebra representations of $\mathfrak{g}$, which is an algebraic problem. Subsequently decide which of the Lie algebra representations of $\mathfrak{g}$ integrate to Lie group representations of G. The latter problem is essentially topological, as it is closely related to the determination of the fundamental group of G. The reduction of the transcendental problem of finding the representations of a Lie group G to a purely algebraic problem of finding the representations of its Lie algebra $\mathfrak{g}$ is the deep and fruitful insight of Sophus Lie.

Exercise 6.1. *Show that for $X, Y \in \mathfrak{gl}_n(\mathbb{R})$ we have*

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X,Y] + O(t^3))$$

for $t \to 0$, by using the convergent power series $\exp(tX) = \sum_0^\infty t^k X^k/k!$. *Conclude that for a homomorphism $\pi : G \to H$ of linear Lie groups the derived homomorphism $\pi : \mathfrak{g} \to \mathfrak{h}$ of their Lie algebras satisfies $\pi([X,Y]) = [\pi(X), \pi(Y)]$ for $X, Y \in \mathfrak{g}$.*

Exercise 6.2. *Suppose G is a connected Lie group. Show that for an open neighborhood U of e in G we have $G = \cup_n U^n$. Conclude that a Lie group homomorphism $\pi : G \to H$ is uniquely determined by its derived Lie algebra $\pi : \mathfrak{g} \to \mathfrak{h}$.*

Exercise 6.3. *Show that a surjective Lie group homomorphism $\pi : G \to H$ with discrete kernel induces a derived Lie algebra isomorphism $\pi : \mathfrak{g} \to \mathfrak{h}$. Conclude that the spin homomorphism π :* $\mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$ *induces a derived Lie algebra isomorphism $\pi : \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}_3(\mathbb{R})$, but the inverse Lie algebra homomorphism $\pi^{-1} : \mathfrak{so}_3(\mathbb{R}) \to \mathfrak{su}_2(\mathbb{C})$ does not integrate to a Lie group homomorphism* $\mathrm{SO}_3(\mathbb{R}) \to \mathrm{SU}_2(\mathbb{C})$.

7 Lie algebra representations

Suppose G_0 is a connected Lie group with *real* Lie algebra $\mathfrak{g}_0$. In the previous section we have seen that each Lie group representation $\rho : G_0 \to \mathrm{GL}(V)$ on a *complex* vector space V can be derived to a Lie algebra representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ on V. Here $\mathfrak{gl}(V) = \mathrm{End}(V)$ as a set, but $\mathrm{End}(V)$ is a *complex* associative algebra with respect to composition of linear maps while $\mathfrak{gl}(V)$ is a *complex* Lie algebra with respect to the commutator bracket. A fortiori $\mathfrak{gl}(V)$ is a real Lie algebra by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$.

Any such homomorphism $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ of real Lie algebras extends to a homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

of complex Lie algebras with $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ the complexification of $\mathfrak{g}_0$. Conversely, any complex representation of the complexification $\mathfrak{g}$ on V restricts to a real representation of $\mathfrak{g}_0$ on V.

The idea is that the connected Lie group G_0 might have a complexification G which should be a connected complex Lie group in the sense that group multiplication and inversion are holomorphic maps. Moreover the smooth representation $\rho : G_0 \to \mathrm{GL}(V)$ might also extend to a holomorphic representation $\rho : G \to \mathrm{GL}(V)$ of the complexification G. Examples show that at the group level this is NOT always possible. However, if such a procedure is possible then the holomorphic representation of G on V is completely determined by the smooth representation of G_0 on V.

Hermann Weyl showed that this procedure is always possible for a *compact* connected Lie group G_0, and in this way he proved that a holomorphic representation of its complexification G on a finite dimensional vector space V is always completely reducible. This method is called *Weyl's unitary trick*. Examples for which the unitary trick works are

G_0	$\mathfrak{g}_0$	$\mathfrak{g}$	G
$\mathrm{U}_1(\mathbb{C}) \cong \mathbb{R}/2\pi\mathbb{Z}$	$\mathfrak{u}_1(\mathbb{C}) \cong \mathbb{R}$	$\mathfrak{gl}_1(\mathbb{C}) \cong \mathbb{C}$	$\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$
$\mathrm{SU}_n(\mathbb{C})$	$\mathfrak{su}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathrm{SL}_n(\mathbb{C})$
$\mathrm{SO}_n(\mathbb{R})$	$\mathfrak{so}_n(\mathbb{R})$	$\mathfrak{so}_n(\mathbb{C})$	$\mathrm{SO}_n(\mathbb{C})$
$\mathrm{Spin}_n(\mathbb{R})$	$\mathfrak{so}_n(\mathbb{R})$	$\mathfrak{so}_n(\mathbb{C})$	$\mathrm{Spin}_n(\mathbb{C})$
$\mathrm{U}_n(\mathbb{H})$	$\mathfrak{u}_n(\mathbb{H})$	$\mathfrak{sp}_{2n}(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C})$

with the easiest and most illuminating case in the second row.

Suppose now that $\mathfrak{g}$ is a complex Lie algebra, and (π, U) and (ρ, V) are finite dimensional representations of $\mathfrak{g}$. By constructions of linear algebra we can define new finite dimensional representations (ρ^*, V^*) on the dual space, $(\pi \otimes \rho, U \otimes V)$ on the tensor product space and $(\mathrm{Hom}(\pi, \rho), \mathrm{Hom}(U, V))$ on the Hom space. Indeed for the dual reprsentation put

$$\rho^*(X) = -\rho(X)^*$$

and for the tensor product representation put

$$(\pi \otimes \rho)(X) = \pi(X) \otimes 1_V + 1_U \otimes \rho(X)$$

with $1_U, 1_V$ the identity operator on U, V respectively, and so for the Hom representation one has to take

$$\mathrm{Hom}(\pi, \rho)(X)A = -A\pi(X) + \rho(X)A$$

for all $X \in \mathfrak{g}$ and $A \in \mathrm{Hom}(U, V)$.

It is an easy exercise in algebra to check that these formulas define new Lie algebra representations. For example, for the tensor product representation this boils down to the verification

$$\begin{gathered}
[(\pi \otimes \rho)(X), (\pi \otimes \rho)(Y)] = \\
(\pi(X) \otimes 1_V + 1_U \otimes \rho(X))(\pi(Y) \otimes 1_V + 1_U \otimes \rho(Y)) - \\
(\pi(Y) \otimes 1_V + 1_U \otimes \rho(Y))(\pi(X) \otimes 1_V + 1_U \otimes \rho(X)) = \\
(\pi(X) \otimes 1_V)(\pi(Y) \otimes 1_V) + (\pi(X) \otimes 1_V)(1_U \otimes \rho(Y)) + \\
(1_U \otimes \rho(X))(\pi(Y) \otimes 1_V) + (1_U \otimes \rho(X))(1_U \otimes \rho(Y)) \\
-X \leftrightarrow Y = \\
(\pi(X)\pi(Y) \otimes 1_V) + (\pi(X) \otimes \rho(Y)) + (\pi(Y) \otimes \rho(X)) + (1_U \otimes \rho(X)\rho(Y)) \\
-X \leftrightarrow Y = \\
(\pi(X)\pi(Y) - \pi(Y)\pi(X)) \otimes 1_V) + (1_U \otimes (\rho(X)\rho(Y) - \rho(Y)\rho(X)) = \\
\pi([X, Y]) \otimes 1_V + 1_U \otimes \rho([X, Y]) = (\pi \otimes \rho)([X, Y])
\end{gathered}$$

for all $X, Y \in \mathfrak{g}$. Likewise for the dual and the Hom representation.

If G_0 is a Lie group with Lie algebra $\mathfrak{g}_0$, and (π, U) and (ρ, V) are finite dimensional representations of G_0 then the defining formula for the tensor product representation also follows from

$$(\pi \otimes \rho)(X) = \frac{d}{dt}\{\pi(\exp(tX)) \otimes \rho(\exp(tX))\}\big|_{t=0} = \pi(X) \otimes 1_V + 1_U \otimes \rho(X)$$

for all $X \in \mathfrak{g}_0$ by using the Leibniz product rule for differentiation. Likewise for the dual and the Hom representation.

Proposition 7.1. *Let V be a representation of a Lie algebra $\mathfrak{g}$. Suppose $H \in \mathfrak{g}$ is a diagonalisable operator in V so that*

$$V = \bigoplus_{\lambda} Ker(H - \lambda)$$

algebraically (any $v \in V$ is a finite sum $v = \sum_{\lambda} v_{\lambda}$ with $Hv_{\lambda} = \lambda v_{\lambda}$). Then any subrepresentation U of V respects this decomposition so that

$$U = \bigoplus_{\lambda} (U \cap Ker(H - \lambda)) \ .$$

Proof. Write $u \in U$ as $u = \sum_{\lambda} u_{\lambda}$ with $Hu_{\lambda} = \lambda u_{\lambda}$. Then $H^n u = \sum_{\lambda} \lambda^n u_{\lambda}$ lie in U for all $n \in \mathbb{N}$. Using the nonvanishing of a Vandermonde determinant we can write each u_{λ} as a linear combination of $H^n u$ with $n \in \mathbb{N}$. $\square$

Exercise 7.1. *Show that the formulas for the dual and Hom representation on the Lie algebra level indeed define Lie algebra representations.*

Exercise 7.2. *Assume that is given that the classical groups $\mathrm{SU}_n(\mathbb{C})$ for $n \geq 2$, $\mathrm{Spin}_n(\mathbb{R})$ for $n \geq 3$ and $\mathrm{U}_n(\mathbb{H})$ for $n \geq 1$ are connected and simply connected. Assume also that is given the theorem that for a connected and simply connected Lie group G_0 with Lie algebra $\mathfrak{g}_0$ each finite dimensional Lie algebra representation of $\mathfrak{g}_0$ integrates to a a Lie group representation of G_0. Conclude that each finite dimensional Lie algebra representation of $\mathfrak{sl}_n(\mathbb{C})$ for $n \geq 2$, or $\mathfrak{so}_n(\mathbb{C})$ for $n \geq 3$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ for $n \geq 1$ is completely reducible.*

8 The universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra of dimension n with a basis $X_1, \cdots, X_n$. The tensor algebra $\mathrm{T}\mathfrak{g} = \oplus \mathrm{T}^d\mathfrak{g}$ is a graded associative algebra with corresponding basis of $\mathrm{T}^d\mathfrak{g}$ given by the pure tensors of degree $d \in \mathbb{N}$

$$\{X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_d}; 1 \leq i_1, i_2, \cdots, i_d \leq n = \dim V\}.$$

The *symmetric algebra* $\mathrm{S}\mathfrak{g} = \oplus \mathrm{S}^d\mathfrak{g} = \mathrm{T}\mathfrak{g}/\mathrm{I}\mathfrak{g} = \mathrm{P}\mathfrak{g}^*$ is also *graded* with degree d part equal to $\mathrm{S}^d\mathfrak{g} = \mathrm{T}^d\mathfrak{g}/\mathrm{I}^d\mathfrak{g}$ with

$$\mathrm{I}^d\mathfrak{g} = \mathrm{I}\mathfrak{g} \cap \mathrm{T}^d\mathfrak{g} = \bigoplus_{k+l=d-2} \bigoplus_{i<j} \{\mathrm{T}^k\mathfrak{g} \otimes (X_i \otimes X_j - X_j \otimes X_i) \otimes \mathrm{T}^l\mathfrak{g}\}.$$

In turn this implies that $\mathrm{S}^d\mathfrak{g}$ has basis

$$\{X_{i_1} X_{i_2} \cdots X_{i_d}; 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n\}$$

or equivalently

$$\{X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}; k_i \in \mathbb{N}\ \forall i, \sum k_i = d\}$$

of monomials of degree d. So far in order to define the symmetric algebra $\mathrm{S}\mathfrak{g}$ we have only used the structure of $\mathfrak{g}$ as a vector space.

Now also the structure of $\mathfrak{g}$ as Lie algebra will come into play. If we denote by $\mathrm{J}\mathfrak{g}$ the (two sided, that is both from the left and the right) ideal of $\mathrm{T}\mathfrak{g}$ defined by

$$\mathrm{J}\mathfrak{g} = \bigoplus_{i<j} \{\mathrm{T}\mathfrak{g} \otimes (X_i \otimes X_j - X_j \otimes X_i - [X_i, X_j]) \otimes \mathrm{T}\mathfrak{g}\}$$

then the *universal enveloping algebra* of the Lie algebra $\mathfrak{g}$ is defined by $\mathrm{U}\mathfrak{g} = \mathrm{T}\mathfrak{g}/\mathrm{J}\mathfrak{g}$. It is an associative algebra, but no longer graded because the ideal $\mathrm{J}\mathfrak{g}$ is no longer homogeneous. What remains is a *filtration* $\mathrm{U}\mathfrak{g} = \cup \mathrm{U}_d\mathfrak{g}$ with

$$\mathrm{T}_d\mathfrak{g} = \bigoplus_{k \leq d} \mathrm{T}^k\mathfrak{g}$$

the tensors of degree at most d, and the intersection

$$\mathrm{J}_d\mathfrak{g} = \mathrm{J}\mathfrak{g} \cap \mathrm{T}_d\mathfrak{g} = \sum_{k+l=d-2} \sum_{i<j} \{\mathrm{T}_k\mathfrak{g} \otimes (X_i \otimes X_j - X_j \otimes X_i - [X_i, X_j]) \otimes \mathrm{T}_l\mathfrak{g}\}$$

of tensors of degree at most d with the ideal, and with quotient

$$\mathrm{U}_d\mathfrak{g} = \mathrm{T}_d\mathfrak{g}/\mathrm{J}_d\mathfrak{g}.$$

Clearly multiplication in $\mathrm{U}\mathfrak{g}$ defines a linear map $\mathrm{U}_d\mathfrak{g} \otimes \mathrm{U}_e\mathfrak{g} \mapsto \mathrm{U}_{d+e}\mathfrak{g}$ for all $d, e \geq 0$.

Definition 8.1. *The* excess e *of a basis vector* $X_{i_1} \otimes \cdots \otimes X_{i_d}$ *of* $\mathrm{T}^d\mathfrak{g}$ *for* $1 \leq i_1, \cdots, i_n \leq n$ *is given by* $e = \# \{(p,q); p < q, i_p > i_q\}$.

We write ${}^e\mathrm{T}^d\mathfrak{g} = \mathrm{span}\{X_{i_1} \otimes \cdots \otimes X_{i_d};\ \text{excess} = e\}$ and ${}^e\mathrm{T}\mathfrak{g} = \oplus_d\, {}^e\mathrm{T}^d\mathfrak{g}$. A monomial $X_{i_1} \otimes \cdots \otimes X_{i_d}$ has excess equal to 0 if and only if $i_1 \leq i_2 \leq \cdots \leq i_d$ in which case we also speak of a standard monomial. The standard monomials form a basis of ${}^0\mathrm{T}$ and descend to the basis $\{X_{i_1} X_{i_2} \cdots X_{i_d}\}$ of $\mathrm{S}\mathfrak{g} = \mathrm{T}\mathfrak{g}/\mathrm{I}\mathfrak{g}$. In other words $\mathrm{T}\mathfrak{g} = {}^0\mathrm{T}\mathfrak{g} \oplus \mathrm{I}\mathfrak{g}$, that is $\mathrm{T}\mathfrak{g} = {}^0\mathrm{T}\mathfrak{g} + \mathrm{I}\mathfrak{g}$ and ${}^0\mathrm{T}\mathfrak{g} \cap \mathrm{I}\mathfrak{g} = 0$.

A similar result holds for the universal enveloping algebra $\mathrm{U}\mathfrak{g} = \mathrm{T}\mathfrak{g}/\mathrm{J}\mathfrak{g}$, and is called the PBW theorem, named after Henri Poincaré, Garrett Birkhoff and Ernst Witt.

Theorem 8.2. *We have* $\mathrm{T}\mathfrak{g} = {}^0\mathrm{T}\mathfrak{g} \oplus \mathrm{J}\mathfrak{g}$.

Proof. We will first show that $\mathrm{T}\mathfrak{g} = {}^0\mathrm{T}\mathfrak{g} + \mathrm{J}\mathfrak{g}$.

Consider a monomial $X_{i_1} \otimes \cdots \otimes X_{i_d} \in {}^e\mathrm{T}^d\mathfrak{g}$ of degree d and excess e. We shall prove that $X_{i_1} \otimes \cdots \otimes X_{i_d} \in {}^0\mathrm{T}\mathfrak{g} + \mathrm{J}\mathfrak{g}$ by a double induction on the degree d and the excess e. If $d = 0$ then $e = 0$, and there is nothing to prove. Likewise if $d \geq 0$ and $e = 0$ then $X_{i_1} \otimes \cdots \otimes X_{i_d} \in {}^0\mathrm{T}\mathfrak{g} \subset {}^0\mathrm{T}\mathfrak{g} + \mathrm{J}\mathfrak{g}$. Now suppose that $e \geq 1$ and say $i_p > i_{p+1}$. Then we write

$$\begin{gathered} X_{i_1} \otimes \cdots \otimes X_{i_d} = X_{i_1} \otimes \cdots \otimes X_{i_p} \otimes X_{i_{p+1}} \otimes \cdots \otimes X_{i_d} = \\ X_{i_1} \otimes \cdots \otimes X_{i_{p+1}} \otimes X_{i_p} \otimes \cdots \otimes X_{i_d} + X_{i_1} \otimes \cdots \otimes [X_{i_p}, X_{i_{p+1}}] \otimes \cdots \otimes X_{i_d} \\ + X_{i_1} \otimes \cdots \otimes (X_{i_p} \otimes X_{i_{p+1}} - X_{i_{p+1}} \otimes X_{i_p} - [X_{i_p}, X_{i_{p+1}}]) \otimes \cdots \otimes X_{i_d} \end{gathered}$$

which lies in ${}^0\mathrm{T}\mathfrak{g} + \mathrm{J}\mathfrak{g}$ by the induction hypothesis, because the first term has degree d and excess $(e-1)$, the second term has degree $(d-1)$ and the third term lies in $\mathrm{J}\mathfrak{g}$.

The second claim ${}^0\mathrm{T}\mathfrak{g} \cap \mathrm{J}\mathfrak{g} = 0$ is proved by a similar induction, but the details are slightly more complicated.

Lemma 8.3. *There exists a (unique) linear map* $L : \mathrm{T}\mathfrak{g} \mapsto \mathrm{T}\mathfrak{g}$ *such that* L *is the identity on* ${}^0\mathrm{T}\mathfrak{g}$ *and (in case* $i_p > i_{p+1}$*)*

$$L(X_{i_1} \otimes \cdots \otimes X_{i_p} \otimes X_{i_{p+1}} \otimes \cdots \otimes X_{i_d}) =$$
$$L(X_{i_1} \otimes \cdots \otimes X_{i_{p+1}} \otimes X_{i_p} \otimes \cdots \otimes X_{i_d}) +$$
$$L(X_{i_1} \otimes \cdots \otimes [X_{i_p}, X_{i_{p+1}}] \otimes \cdots \otimes X_{i_d}) \ .$$

We shall refer to this formula as the ordering formula. Note that the ordering formula amounts to

$$L(X_{i_1} \otimes \cdots \otimes (X_{i_p} \otimes X_{i_{p+1}} - X_{i_{p+1}} \otimes X_{i_p} - [X_{i_p}, X_{i_{p+1}}]) \otimes \cdots \otimes X_{i_d}) = 0$$

so that $L = 0$ on $\mathrm{J}\mathfrak{g}$. Hence ${}^0\mathrm{T}\mathfrak{g} \cap \mathrm{J}\mathfrak{g} \subset \ker(L-1) \cap \ker(L) = 0$. Therefore the second claim is a direct consequence of the existence of the linear map L of the lemma.

Proof. We prove the lemma by a double induction on the degree d and the excess e. The problem is to define L on the monomial basis $X_{i_1} \otimes \cdots \otimes X_{i_d}$ of ${}^e\mathrm{T}^d\mathfrak{g}$. If $e = 0$ then $L(X_{i_1} \otimes \cdots \otimes X_{i_d}) = X_{i_1} \otimes \cdots \otimes X_{i_d}$ by assumption. If there is unique index $1 \leq p < d$ such that $i_p > i_{p+1}$ then we put

$$L(X_{i_1} \otimes \cdots \otimes X_{i_p} \otimes X_{i_{p+1}} \otimes \cdots \otimes X_{i_d}) =$$
$$L(X_{i_1} \otimes \cdots \otimes X_{i_{p+1}} \otimes X_{i_p} \otimes \cdots \otimes X_{i_d}) +$$
$$L(X_{i_1} \otimes \cdots \otimes [X_{i_p}, X_{i_{p+1}}] \otimes \cdots \otimes X_{i_d})$$

and the result is well defined by the induction hypothesis.

If $i_p > i_{p+1}$ and $i_q > i_{q+1}$ for some $p < q$ then we rewrite

$$I = L(X_{i_1} \otimes \cdots \otimes X_{i_p} \otimes X_{i_{p+1}} \otimes \cdots \otimes X_{i_q} \otimes X_{i_{q+1}} \otimes \cdots \otimes X_{i_d})$$

using the the ordering formula of the lemma as a linear combination (involving the structure constants of $\mathfrak{g}$ for the given basis) of the image under L of monomials of lower excess or lower degree. However the rewriting can be done using the ordering formula at place $(p, p+1)$ or at place $(q, q+1)$. We have to check that the outcome is the same, and so the linear map L is well defined.

Let us first assume that $p+1 < q$. Let the final expression F be obtained from I by using the ordering formula at place $(p, p+1)$, and let F' be obtained from I by using the ordering formula at place $(q, q+1)$. Then we get $F = F'$

by using the above lemma for F at place $(q, q+1)$ and for F' at place $(p, p+1)$, which is allowed by induction on the excess and the degree. Explicitly

$$\begin{aligned} F &= L(X_{i_1} \otimes \cdots \otimes X_{i_{p+1}} \otimes X_{i_p} \otimes \cdots \otimes X_{i_q} \otimes X_{i_{q+1}} \otimes \cdots \otimes X_{i_d}) \\ &+ L(X_{i_1} \otimes \cdots \otimes [X_{i_p}, X_{i_{p+1}}] \otimes \cdots \otimes X_{i_q} \otimes X_{i_{q+1}} \otimes \cdots \otimes X_{i_d}) \end{aligned}$$

which can be rewritten by the induction hypothesis as

$$\begin{aligned} F &= L(X_{i_1} \otimes \cdots \otimes X_{i_{p+1}} \otimes X_{i_p} \otimes \cdots \otimes X_{i_{q+1}} \otimes X_{i_q} \otimes \cdots \otimes X_{i_d}) \\ &+ L(X_{i_1} \otimes \cdots \otimes X_{i_{p+1}} \otimes X_{i_p} \otimes \cdots \otimes [X_{i_q}, X_{i_{q+1}}] \otimes \cdots \otimes X_{i_d}) \\ &+ L(X_{i_1} \otimes \cdots \otimes [X_{i_p}, X_{i_{p+1}}] \otimes \cdots \otimes X_{i_{q+1}} \otimes X_{i_q} \otimes \cdots \otimes X_{i_d}) \\ &+ L(X_{i_1} \otimes \cdots \otimes [X_{i_p}, X_{i_{p+1}}] \otimes \cdots \otimes [X_{i_q}, X_{i_{q+1}}] \otimes \cdots \otimes X_{i_d}) . \end{aligned}$$

The outcome has a symmetric role in p and q, and hence $F = F'$.

Next assume that $p + 1 = q$. Our initial expression becomes

$$I = L(X_{i_1} \otimes \cdots \otimes X_{i_p} \otimes X_{i_{p+1}} \otimes X_{i_{p+2}} \otimes \cdots \otimes X_{i_d})$$

or more simply (but without loss of generality)

$$I = L(X_{i_1} \otimes X_{i_2} \otimes X_{i_3})$$

with $i_1 > i_2 > i_3$. So we have

$$\begin{gathered} F = L(X_{i_2} \otimes X_{i_1} \otimes X_{i_3}) + L([X_{i_1}, X_{i_2}] \otimes X_{i_3}) = \\ L(X_{i_2} \otimes X_{i_3} \otimes X_{i_1}) + L(X_{i_2} \otimes [X_{i_1}, X_{i_3}]) + L([X_{i_1}, X_{i_2}] \otimes X_{i_3}) = \\ L(X_{i_3} \otimes X_{i_2} \otimes X_{i_1}) + L([X_{i_2}, X_{i_3}] \otimes X_{i_1}) + \\ L(X_{i_2} \otimes [X_{i_1}, X_{i_3}]) + L([X_{i_1}, X_{i_2}] \otimes X_{i_3}) \end{gathered}$$

and similarly

$$\begin{gathered} F' = L(X_{i_1} \otimes X_{i_3} \otimes X_{i_2}) + L(X_{i_1} \otimes [X_{i_2}, X_{i_3}]) = \\ L(X_{i_3} \otimes X_{i_1} \otimes X_{i_2}) + L([X_{i_1}, X_{i_3}] \otimes X_{i_2}) + L(X_{i_1} \otimes [X_{i_2}, X_{i_3}]) = \\ L(X_{i_3} \otimes X_{i_2} \otimes X_{i_1}) + L(X_{i_3} \otimes [X_{i_1}, X_{i_2}]) + \\ L([X_{i_1}, X_{i_3}] \otimes X_{i_2}) + L(X_{i_1} \otimes [X_{i_2}, X_{i_3}]) . \end{gathered}$$

Hence we find

$$\begin{gathered} F - F' = L([X_{i_2}, X_{i_3}] \otimes X_{i_1} - X_{i_1} \otimes [X_{i_2}, X_{i_3}]) + \\ L(X_{i_2} \otimes [X_{i_1}, X_{i_3}] - [X_{i_1}, X_{i_3}] \otimes X_{i_2}) + \\ L([X_{i_1}, X_{i_2}] \otimes X_{i_3} - X_{i_3} \otimes [X_{i_1}, X_{i_2}]) = \\ L([[X_{i_2}, X_{i_3}], X_{i_1}] + [X_{i_2}, [X_{i_1}, X_{i_3}]] + [[X_{i_1}, X_{i_2}], X_{i_3}]) \\ L([[X_{i_2}, X_{i_3}], X_{i_1}] + [[X_{i_3}, X_{i_1}], X_{i_2}] + [[X_{i_1}, X_{i_2}], X_{i_3}]) = L(0) = 0 \end{gathered}$$

by the antisymmetry of the bracket and the Jacobi identity. This concludes the proof that $F = F'$ in the second case, and finishes the proof of the lemma. $\square$

As mentioned before the ordering formula of the lemma has the theorem as immediate consequence. This finishes the proof of the PBW theorem. $\square$

Corollary 8.4. *Given a basis $X_1, \cdots, X_n$ of $\mathfrak{g}$ the universal enveloping algebra $\mathrm{U}\mathfrak{g}$ has the corresponding PBW basis*

$$\{X_{i_1} X_{i_2} \cdots X_{i_d};\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n,\ d \in \mathbb{N}\}\ ,$$

or equivalently

$$\{X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n};\ 0 \leq k_1, k_2, \cdots, k_n < \infty\}\ .$$

Corollary 8.5. *The natural map $j : \mathfrak{g} \to \mathrm{U}\mathfrak{g}$ is an injection. From now on we write $X \mapsto X$ for the canonical injection $\mathfrak{g} \hookrightarrow \mathrm{U}\mathfrak{g}$.*

This corollary justifies the terminology universal enveloping algebra for $\mathrm{U}\mathfrak{g}$, since it is truly an enveloping associative algebra of $\mathfrak{g}$. Moreover $\mathrm{U}\mathfrak{g}$ is generated as an associative algebra by $\mathfrak{g}$. Finally it is universal with respect to these two properties.

Exercise 8.1. *Show that the natural linear bijection $\mathrm{T}_d\mathfrak{g}/\mathrm{T}_{d-1}\mathfrak{g} \to \mathrm{T}^d\mathfrak{g}$ induces a linear bijection $\mathrm{U}_d\mathfrak{g}/\mathrm{U}_{d-1}\mathfrak{g} \to \mathrm{S}^d\mathfrak{g}$ that is compatible with the multiplications in $\mathrm{U}\mathfrak{g}$ and $\mathrm{S}\mathfrak{g}$. Show that the Poisson bracket $\{\cdot,\cdot\}$*

$$\mathrm{S}^d\mathfrak{g} \otimes \mathrm{S}^e\mathfrak{g} \to \mathrm{S}^{d+e-1}\mathfrak{g}$$

with $\{\cdot,\cdot\} = [\cdot,\cdot]$ for $d = e = 1$ and $\{x, yz\} = \{x, y\}z + y\{x, z\}$, amounts to

$$\mathrm{U}_d\mathfrak{g}/\mathrm{U}_{d-1}\mathfrak{g} \otimes \mathrm{U}_e\mathfrak{g}/\mathrm{U}_{e-1}\mathfrak{g} \to \mathrm{U}_{d+e-1}\mathfrak{g}/\mathrm{U}_{d+e-2}\mathfrak{g}$$

sending the quotients of $x \otimes y$ to the quotient of $xy - yx$ (multiplication in $U\mathfrak{g}$). The conclusion is that the Poisson bracket is the leading term of the commutator bracket. This is analogous to the way classical mechanics can be obtained from quantum mechanics as the classical limit.

9 The representations of $\mathfrak{sl}_2(\mathbb{C})$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has so called Chevalley basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutation relations

$$[H, E] = 2E \ , \ [H, F] = -2F \ , \ [E, F] = H \ .$$

The universal enveloping algebra $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$ has the associated PBW basis $F^i H^j E^k$ for $i, j, k \in \mathbb{N}$.

Lemma 9.1. *In* $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$ *the following relations hold*

$$[H, F^{k+1}] = -2(k+1)F^{k+1} \quad , \quad [E, F^{k+1}] = (k+1)F^k(H-k)$$

for all $k \in \mathbb{N}$.

Proof. In any associative algebra $\mathcal{A}$ we just write $[a, b] = ab - ba$ (turning $\mathcal{A}$ also into a Lie algebra) and we have the trivial formula $[a, bc] = [a, b]c + b[a, c]$ for all $a, b, c \in \mathcal{A}$.

The proof of the first formula goes by induction on $k \in \mathbb{N}$. The case $k = 0$ is clear, since $[H, F] = -2F$. Now for $k \geq 1$ we have

$$[H, F^{k+1}] = [H, FF^k] = [H, F]F^k + F[H, F^k] = \\ -2FF^k - 2kFF^k = -2(k+1)F^{k+1}$$

which proves the first formula.

Likewise the proof of the second formula goes by induction on $k \in \mathbb{N}$. The case $k = 0$ is clear, since $[E, F] = H$. Now for $k \geq 1$ we have

$$[E, F^{k+1}] = [E, F]F^k + F[E, F^k] = HF^k + kFF^{k-1}(H - (k-1))$$

by using the induction hypothesis. Using the first formula we get

$$[E, F^{k+1}] = [H, F^k] + F^k H + kF^k(H - (k-1)) = \\ -2kF^k + F^k H + kF^k(H - (k-1))$$

and therefore

$$[E, F^{k+1}] = (k+1)F^k H - F^k(2k + k(k-1)) = (k+1)F^k(H-k)$$

which proves the second formula. □

The universal enveloping algebra $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$ has a natural (associative algebra) representation on the infinite dimensional vector space $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$ by left multiplication. For $\lambda \in \mathbb{C}$ the linear subspace

$$\mathrm{J}(\lambda) = \mathrm{U}\mathfrak{sl}_2(\mathbb{C})(H - \lambda) + (\mathrm{U}\mathfrak{sl}_2)(\mathbb{C})E$$

is a left ideal in $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$, or equivalently an invariant linear subspace of $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$. Hence the quotient

$$M(\lambda) = \mathrm{U}\mathfrak{sl}_2(\mathbb{C})/\mathrm{J}(\lambda)$$

also becomes an associative algebra representation space of $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$, called the Verma representation of $\mathrm{U}\mathfrak{sl}_2(\mathbb{C})$ with parameter $\lambda \in \mathbb{C}$. By restriction from $\mathrm{Usl}_2(\mathbb{C})$ to $\mathrm{sl}_2(\mathbb{C})$ the Verma representation $M(\lambda)$ can be equally well considered as a Lie algebra representation of $\mathfrak{sl}_2(\mathbb{C})$.

Lemma 9.2. *The Verma representation $M(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$ has basis*

$$v_k = F^k + \mathrm{J}(\lambda)$$

for $k \in \mathbb{N}$, and the generators $\{E, H, F\}$ act on this basis as

$$Hv_k = (\lambda - 2k)v_k \ , \ Ev_k = k(\lambda - (k-1))v_{k-1} \ , \ Fv_k = v_{k+1}$$

with $v_{-1} = 0$. The operator H in $M(\lambda)$ is diagonal in the basis v_k for $k \in \mathbb{N}$. The operators E and F in $M(\lambda)$ are so called ladder *operators.*

Proof. The PBW basis vector $F^k H^l E^m$ can be reduced modulo $J(\lambda)$ to a multiple of F^k. Hence $v_k,\ k \in \mathbb{N}$ span $M(\lambda)$, and $v_k \neq 0\ \forall k \in \mathbb{N}$.

By definition $v_0 = 1 + J(\lambda)$, hence

$$Hv_0 \equiv H = \lambda + (H - \lambda) \equiv \lambda \equiv \lambda v_0$$

with $\equiv$ denoting equality modulo $J(\lambda)$. Moreover

$$Hv_k = HF^k v_0 = [H, F^k]v_0 + F^k H v_0 = -2kF^k v_0 + \lambda F^k v_0 = (\lambda - 2k)v_k$$

and therefore $v_k,\ k \in \mathbb{N}$ is a basis of $M(\lambda)$. Clearly

$$Ev_0 \equiv E \equiv 0$$

so that

$$Ev_k = EF^k v_0 = [E, F^k]v_0 + F^k E v_0 = k(\lambda - (k-1))v_{k-1} \ .$$

The last formula $Fv_k = v_{k+1}$ is trivial. $\square$

Definition 9.3. *The eigenvalues of the operator H in a representation space of $\mathfrak{sl}_2$ are called the* weights *of that representation. For example, the weights of the Verma representation $M(\lambda)$ are $\{\lambda - 2k;\ k \in \mathbb{N}\}$ and the parameter $\lambda \in \mathbb{C}$ is therefore called the* highest weight *of $M(\lambda)$. The vector v_0 is called the highest weight vector in $M(\lambda)$.*

Lemma 9.4. *A linear subspace of $M(\lambda)$ that is invariant under H is the linear span of some of the v_k. A nonzero linear subspace of $M(\lambda)$ that is invariant under H and F is the linear span of the v_k for $k \geq k_0$, for some $k_0 \in \mathbb{N}$.*

Proof. The first statement follows from Proposition 7.1, and the second statement is obvious from the first statement. □

Corollary 9.5. *A nontrivial (distinct from 0 and the full space) linear subspace of $M(\lambda)$ that is invariant under $\mathfrak{sl}_2$ exists if an only if $Ev_{n+1} = 0$ for some $n \in \mathbb{N}$,and so $\lambda = n \in \mathbb{N}$. In this case v_{n+1} is a highest weight vector of weight $-n-2$ and we get an intertwining operator $M(-n-2) \hookrightarrow M(n)$ with irreducible quotient representation $L(n) = M(n)/M(-n-2)$ of dimension $(n+1)$ and with weights $\{n, n-2, \cdots, -n\}$.*

We shall denote by $L(\lambda)$ the irreducible quotient of $M(\lambda)$. In other words $L(\lambda) = M(n)/M(-n-2)$ if $\lambda = n \in \mathbb{N}$ while $L(\lambda) = M(\lambda)$ if $\lambda \notin \mathbb{N}$.

Remark 9.6. *Let $V = \mathbb{C}^2$ be a two dimensional vector space with basis $\{X, Y\}$. The vector space $V_n = S^nV$ of binary forms of degree n has basis $\{X^n, X^{n-1}Y, \cdots, XY^{n-1}, Y^n\}$. The standard representation ρ of $SL_2(\mathbb{C})$ of V gives rise to an irreducible representation $S^n\rho$ on V_n as mentioned in Remark 5.5. The corresponding Lie algebra representation of $\mathfrak{sl}_2(\mathbb{C})$ on $V_n = L(n)$ is given by the linear vector fields*

$$E = X\frac{\partial}{\partial Y} \quad , \quad F = Y\frac{\partial}{\partial X} \quad , \quad H = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$$

with highest weight vector $v_0 = X^n$ of weight $n \in \mathbb{N}$.

Let $\mathfrak{g}_0$ be a real Lie algebra with complexification $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. If we denote $(X + iY)^\star = -X + iY$ for $X, Y \in \mathfrak{g}_0$ then the star is an *antilinear antiinvolution* on $\mathfrak{g}$. The star extends to $\mathrm{U}\mathfrak{g}$ as an antilinear antiinvolution. If $\langle\cdot,\cdot\rangle$ is a Hermitian inner product on a vector space V then a representation of $\mathfrak{g}_0$ on on $(V, \langle\cdot,\cdot\rangle)$ is called unitary if $\mathfrak{g}_0$ acts via skew Hermitian linear operators on V or equivalently in the representation space

$$X^* = X^\dagger \quad , \quad x^* = x^\dagger$$

for all $X \in \mathfrak{g}$ or all $x \in \mathrm{U}\mathfrak{g}$ with the dagger the adjoint operator on $(V, \langle\cdot,\cdot\rangle)$.

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has two real forms $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ with corresponding antilinear antiinvolutions

$$H^\star = H \ , \ E^\star = F \ , \ F^\star = E$$

$$H^\star = H \ , \ E^\star = -F \ , \ F^\star = -E$$

respectively. A Hermitian form $\langle\cdot,\cdot\rangle$ on V assigns to each pair $u, v \in V$ a complex number $\langle u, v\rangle$ which is linear in $u \in V$ and antilinear in $v \in V$. We do not require the Hermitian form to be unitary (which means $\langle v, v\rangle$ positive for all $v \neq 0$), and even the kernel of the Hermitian form (which consists of all $u \in V$ with $\langle u, v\rangle = 0$ for all $v \in V$) might be a nonzero linear subspace. Now let V also be a representation space for $\mathfrak{sl}_2(\mathbb{C})$. The Hermitian form $\langle\cdot,\cdot\rangle$ is called invariant for $\mathfrak{sl}_2(\mathbb{C})$ with respect to the given star structure if

$$\langle Xu, v\rangle = \langle u, X^\star v\rangle \ \forall u, v \in V \ , \ \forall x \in \mathfrak{sl}_2(\mathbb{C}) \ .$$

In particular for $V = M(\lambda)$ an invariant Hermitian form $\langle\cdot,\cdot\rangle$ satisfies

$$(\lambda - 2k)\langle v_k, v_l\rangle = \langle Hv_k, v_l\rangle = \langle v_k, Hv_l\rangle = (\overline{\lambda} - 2l)\langle v_k, v_l\rangle$$

and hence $((\lambda - \overline{\lambda}) - 2(k-l))\langle v_k, v_l\rangle = 0$. Since $(\lambda - \overline{\lambda}) \in i\mathbb{R}$ and $(k-l) \in \mathbb{Z}$ we get $\langle v_k, v_l\rangle = 0$ if $k \neq l$. Furthermore we find

$$\langle v_{k+1}, v_{k+1}\rangle = \langle v_{k+1}, Fv_k\rangle = \pm\langle Ev_{k+1}, v_k\rangle = \pm(k+1)(\lambda - k)\langle v_k, v_k\rangle$$

with $\pm = +$ for $\mathfrak{su}(2)$ and $\pm = -$ for $\mathfrak{su}(1,1)$.

Theorem 9.7. *The irreducible representation $L(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$ is unitary for $\mathfrak{su}(2)$ if and only if $\lambda = n \in \mathbb{N}$, that is $L(\lambda = n)$ has finite dimension $(n+1)$.*

Proof. We may assume that $\langle v_0, v_0\rangle$ is positive. If $\lambda = n \in \mathbb{N}$ then in $M(n)$ we see that

$$\langle v_k, v_k\rangle = (k!)^2 \binom{n}{k} \langle v_0, v_0\rangle$$

is positive for $0 \leq k \leq n$, while $\langle v_{k+1}, v_{k+1} \rangle = 0 \; \forall k \geq n$. Hence the kernel of the invariant Hermitian form on $M(n)$ is exactly equal to the maximal proper subrepresentation $M(-n-2)$, and the invariant Hermitian form on $M(n)$ descends to an invariant unitary structure on $L(n)$. If $\lambda \notin \mathbb{N}$ then $M(\lambda) = L(\lambda)$ is a unitary representation for $\mathfrak{su}(2)$ if and only if $(\lambda - k) > 0 \; \forall k \in \mathbb{N}$. These inequalities have no solution. $\square$

All unitary representations $L(n)$ for $n \in \mathbb{N}$ of $\mathfrak{su}(2)$ can be integrated to unitary representations of the group SU(2). In fact these are all the unitary irreducible representations of SU(2).

Theorem 9.8. *The irreducible representation $L(\lambda)$ of $\mathfrak{sl}_2$ is unitary for $\mathfrak{su}(1,1)$ if and only if $\lambda \leq 0$.*

Proof. Assume that $\langle v_0, v_0 \rangle$ is positive. Like in the above proof we get

$$\langle v_k, v_k \rangle = (k!)^2 (-1)^k \binom{\lambda}{k} \langle v_0, v_0 \rangle \; .$$

Clearly $\langle v_1, v_1 \rangle = (-\lambda)\langle v_0, v_0 \rangle$ to be nonnegative is a necessary condition for unitarity of $L(\lambda)$, that is $\lambda \leq 0$. If $\lambda = 0$ then $L(0)$ is the one dimensional trivial representation, which is always unitary. If $\lambda < 0$ then $L(\lambda) = M(\lambda)$ is indeed unitary since $-(\lambda - k) > 0 \; \forall k \in \mathbb{N}$. $\square$

All unitary representations $L(-n)$ for $n \in \mathbb{N}$ of $\mathfrak{su}(1,1)$ can be integrated (after a Hilbert space completion) to unitary representations of the group $\mathrm{SU}(1,1)$. For $\lambda = -n < -1$ these representations are so called "discrete series" representations of $\mathrm{SU}(1,1)$, because all the matrix coëfficients are square integrable with respect to a biinvariant measure on $\mathrm{SU}(1,1)$. These (and their lowest weight companions in the first exercise below) are not all the unitary irreducible representations of $\mathrm{SU}(1,1)$. The family of unitary representations of $\mathfrak{su}(1,1)$ described in the above theorem is ususally referred to as the analytic continuation of the discrete series representations of $\mathrm{SU}(1,1)$.

Exercise 9.1. *The Verma representations $M(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$ are sometimes called the highest weight Verma representations to distinguish them from the lowest weight Verma representations $M'(\lambda) = U\mathfrak{sl}_2/J'(\lambda)$ defined by*

$$J'(\lambda) = U\mathfrak{sl}_2(\mathbb{C})(H - \lambda) + U\mathfrak{sl}_2(\mathbb{C})F \; .$$

Sow that for $n \in \mathbb{N}$ the irreducible quotient $L'(-n)$ has finite dimension $(n+1)$, and is unitary for $\mathfrak{su}(2)$. Show that $L(n)$ and $L'(-n)$ are equivalent representations. Show that the irreducible quotient $L'(\lambda)$ is unitary for $\mathfrak{su}(1,1)$ in case $\lambda \geq 0$.

Exercise 9.2. *Write formally $\exp(X) = \sum_{n\geq 0} X^n/n!$ for the exponential series.*

1. *Show that $\exp(X)\exp(Y) = \exp(X+Y)$ if $XY = YX$.*
2. *Deduce that $\exp(X)$ is invertible with inverse $\exp(-X)$.*
3. *Show that $\exp(E)$ is a well defined operator in the Verma representation $M(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$.*
4. *Show that $\exp(F)$ is a well defined operator in the finite dimensional representation $L(n)$, $n \in \mathbb{N}$ of $\mathfrak{sl}_2(\mathbb{C})$.*
5. *Show that in the adjoint representation*
$$\mathrm{ad} : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(\mathfrak{sl}_2(\mathbb{C})),\ \mathrm{ad}(X)Y = [X,Y]$$
the operator $s = \exp(E)\exp(-F)\exp(E)$ satisfies
$$sH = -H\ ,\ sE = -F\ ,\ sF = -E\ .$$
6. *Show that $(\exp(X))Y(\exp(-X)) = \exp(\mathrm{ad}(X))y$ in any finite dimensional representation space of a Lie algebra $\mathfrak{g}$.*
7. *Conclude that the operator $s = \exp(E)\exp(-F)\exp(E)$ is well defined for any finite dimensional representation $\mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(V)$ and maps vectors of weight k to vectors of weight $-k$.*

Exercise 9.3. *If U and V are unitary representations of $\mathfrak{g}$ then check that $U \otimes V$ and V^* are unitary representations of $\mathfrak{g}$ as well.*

Exercise 9.4. *Show that the degree 2 invariant in $\mathrm{S}(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$ is given by the expression $H^2 + 4EF$.*

Exercise 9.5. *Show that the Casimir operator $C = H^2 + 2EF + 2fE$ is central in $\mathrm{U}\mathfrak{sl}_2$. By Schur's Lemma the Casimir operator C acts in $L(n)$ as a scalar operator. Rewriting $C = H^2 + 2H + 4FE$ check that this scalar equals $n^2 + 2n = (n+1)^2 - 1$.*

Exercise 9.6. *Show that the Casimir operator C in a unitary representation V of $\mathfrak{su}_2$ is selfadjoint. The eigenspace decomposition $V = \oplus V(n)$ with $V(n) = Ker(C - (n^2 + 2n))$ is called the direct sum decomposition of V in isotypical components, shortly the isotypical decomposition. Apparently the direct sum decomposition of V in irreducible components is not unique, but the isotypical decomposition of V is unique by the above argument.*

Exercise 9.7. *Suppse we have given a representation of $\mathfrak{sl}_2$ on a finite dimensional Hilbert space V that is unitary for $\mathfrak{su}_2$. The eigenvalues of H on V are called the weights of the representation on V. Suppose $k \in \mathbb{Z}$ occurs as weight of the representaion on V with multiplicity $m_k \in \mathbb{N}$.*

1. *Show that $m_{-k} = m_k$ for all $k \in \mathbb{Z}$.*
2. *Show that $m_0 \geq m_2 \geq m_4 \geq \cdots$ and $m_1 \geq m_3 \geq m_5 \geq \cdots$. The first and second items together are rephrased by saying that the multiplicities of the even weights and of the odd weights form so called palindromic sequences.*
3. *Suppose all weights of V are even and $m_0 = 5, m_2 = 3, m_4 = 3, m_8 = 2, m_{12} = 1, m_{14} = 0$. Describe the decomposition of V into irreducible components.*
4. *Suppose all weights of V are odd and $m_5 = 4, m_7 = 1$ and the dimension of V is equal to 28. Describe the decomposition of V into irreducible components. These data do not quite suffice for a unique solution, but there are two possibilities.*

10 The quantization of the Kepler problem

In this section we will discuss the Kepler problem from the viewpoint of both classical mechanics (planetary motion around the sun) and quantum mechanics (the hydrogen atom). Our discussion of the classical mechanics of the Kepler problem is taken from an article "Teaching the Kepler laws for freshmen" by Maris van Haandel and myself from 2009 and published in Mathematical Intelligencer.

We shall use inner products $\mathbf{u}\cdot\mathbf{v}$ and outer products $\mathbf{u}\times\mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the compatibility conditions

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

and the Leibniz product rules

$$(\mathbf{u}\cdot\mathbf{v})^{\cdot} = \dot{\mathbf{u}}\cdot\mathbf{v} + \mathbf{u}\cdot\dot{\mathbf{v}}$$

$$(\mathbf{u}\times\mathbf{v})^{\cdot} = \dot{\mathbf{u}}\times\mathbf{v} + \mathbf{u}\times\dot{\mathbf{v}}$$

without further explanation.

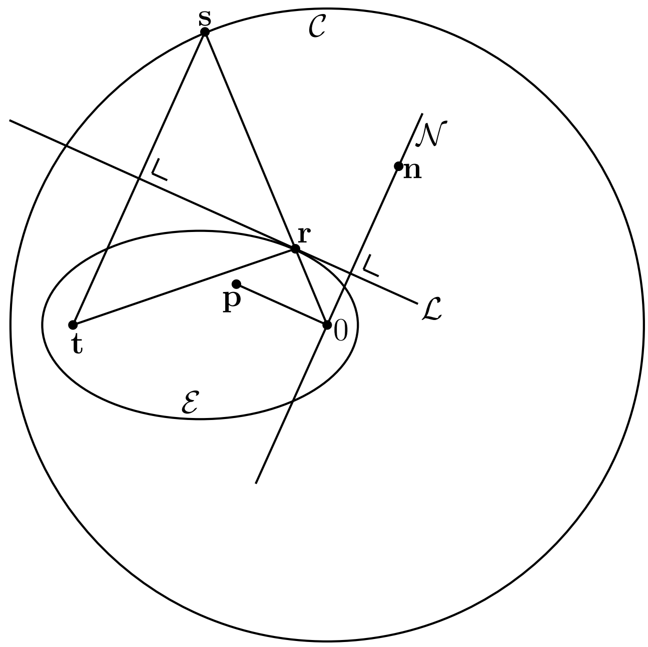
For a central force field $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{r}/r$ the angular momentum vector $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is conserved by Newton's law of motion $\mathbf{F} = \dot{\mathbf{p}}$, thereby leading to Kepler's second law. For a spherically symmetric central force field $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}/r$ the energy

$$H = p^2/2m + V(r)\ ,\ V(r) = -\int f(r)\,dr$$

is conserved as well. These are the general initial remarks.

From now on consider the Kepler problem $f(r) = -k/r^2$ en $V(r) = -k/r$ with $k > 0$ a coupling constant. More precisely, we consider the reduced Kepler problem for the sun S and a planet P with $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_S$ the relative position of the planet. Let m_S and m_P be the masses of the sun S and the planet P respectively. According to Newton the reduced mass becomes $m = m_S m_P/(m_S + m_P)$ while the coupling constant is given by $k = G m_S m_P$ with G the universal gravitational constant. By conservation of energy the motion for fixed energy $H < 0$ is bounded inside a sphere with center 0 and radius $-k/H$.

Consider the following picture of the plane perpendicular to $\mathbf{L}$. The circle $\mathcal{C}$ with center 0 and radius $-k/H$ is the boundary of a disc where motion with energy $H < 0$ takes place. Let $\mathbf{s} = -k\mathbf{r}/rH$ be the projection of $\mathbf{r}$ from the center 0 on this circle $\mathcal{C}$. The line $\mathcal{L}$ through $\mathbf{r}$ with direction vector $\mathbf{p}$ is the tangent line of the orbit $\mathcal{E}$ at position $\mathbf{r}$ with velocity $\mathbf{v}$. Let $\mathbf{t}$ be the orthogonal reflection of the point $\mathbf{s}$ in the line $\mathcal{L}$. As time varies, the position vector $\mathbf{r}$ moves along the orbit $\mathcal{E}$, and likewise $\mathbf{s}$ moves along the circle $\mathcal{C}$. It is a good question to investigate how the point $\mathbf{t}$ moves.



Theorem 10.1. *The point* $\mathbf{t}$ *equals* $\mathbf{K}/mH$ *and therefore is conserved.*

Proof. The line $\mathcal{N}$ spanned by $\mathbf{n} = \mathbf{p} \times \mathbf{L}$ is perpendicular to $\mathcal{L}$. The point $\mathbf{t}$ is obtained from $\mathbf{s}$ by subtracting twice the orthogonal projection of $\mathbf{s} - \mathbf{r}$ on the line $\mathcal{N}$, and therefore

$$\mathbf{t} = \mathbf{s} - 2((\mathbf{s} - \mathbf{r}) \cdot \mathbf{n})\mathbf{n}/n^2.$$

Now

$$\mathbf{s} = -k\mathbf{r}/rH$$

$$(\mathbf{s} - \mathbf{r}) \cdot \mathbf{n} = -(H + k/r)\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L})/H = -(H + k/r)L^2/H$$

$$n^2 = p^2L^2 = 2m(H + k/r)L^2$$

and therefore

$$\mathbf{t} = -k\mathbf{r}/rH + \mathbf{n}/mH = \mathbf{K}/mH$$

with $\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r$ the Lenz vector. The final step $\dot{\mathbf{K}} = 0$ is derived by a straightforward computation using the compatibility relations and the Leibniz product rules for inner and outer products of vectors in $\mathbb{R}^3$. $\square$

Corollary 10.2. *The orbit $\mathcal{E}$ is an ellipse with foci 0 and $\mathbf{t}$, and major axis equal to $2a = -k/H$.*

Proof. Indeed we have

$$|\mathbf{t} - \mathbf{r}| + |\mathbf{r} - 0| = |\mathbf{s} - \mathbf{r}| + |\mathbf{r} - 0| = |\mathbf{s} - 0| = -k/H.$$

Hence $\mathcal{E}$ is an ellipse with foci 0 and $\mathbf{t}$, and major axis $2a = -k/H$. $\square$

The conserved vector $\mathbf{t} = \mathbf{K}/mH$ is a priori well motivated in Euclidean geometric terms. In most text books on classical mechanics (e.g. H. Goldstein, Classical Mechanics) the Lenz vector K is written down, and the motivation comes only a posteriori from $\dot{\mathbf{K}} = \mathbf{0}$ as a vector in the direction of the long axis of the elliptical orbit. The Lenz vector goes already back to Lagrange in his article "Théorie des variations séculaires des élements des planètes" from 1781.

It is easy to check the relations $\mathbf{K}\cdot\mathbf{L} = 0$ and $K^2 = 2mHL^2 + k^2m^2$. Hence besides the familiar conserved quantities angular momentum $\mathbf{L}$ and energy H only the direction of the Lenz vector $\mathbf{K}$ is a new independent conserved quantity. Altogether there are $3+1+1 = 5$ independent conserved quantities, whose level curves in the phase space $\mathbb{R}^6$ are the Kepler ellipses, at least for $H < 0$ corresponding to bounded motion.

If f, g are smooth functions of the six independent coordinates $\mathbf{q} = (q_1, q_2, q_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$ and if we define the *Poisson bracket* $\{f, g\}$ by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial r_i} - \frac{\partial f}{\partial r_i} \frac{\partial g}{\partial p_i} \right)$$

then it is easy to check that

$$\{f, g\} = -\{g, f\} \ \ , \ \ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

for all three functions f, g, h. Hence the linear space $\mathcal{F}$ of smooth functions of $\mathbf{r}$ and $\mathbf{p}$ have the structure of a Lie algebra. In addition the Poisson bracket relates to the (commutative!) pointwise multiplication of functions by

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

for all three functions $f, g, h \in \mathcal{F}$.

Newton's equation of motion for a conservative force field $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ can be written in *Hamiltonian form*

$$\dot{f} = \{H, f\}$$

with $H = p^2/(2m) + V(\mathbf{r})$ the Hamiltonian. Indeed, for $f = r_i$ or $f = p_i$ Hamilton's equation amounts to

$$m\dot{\mathbf{r}} = \mathbf{p} \ \ , \ \ \dot{\mathbf{p}} = \mathbf{F}(\mathbf{r})$$

which is just Newton's equation.

We now turn to the quantum mechanics of the Kepler problem. The first discussion of this question was given by Pauli in an article "On the hydrogen spectrum from the standpoint of the new quantum mechanics" from January 1926. Pauli's solution is a beautiful piece of Lie algebra theory, which made it difficult to digest for the average physicist of that time, who happily adopted the shortly after found solution by Schrödinger. The latter method rewrites the Schrödinger eigenvalue equation in spherical coördinates, and subsequently consults a book on Special Functions (for Laguerre and Legendre polynomials). This method of Schrödinger is just a lot of calculations. In the proof by Pauli one has to do also a fair amount of calculations. But the proof by Pauli is by far the better, because it explains the natural degeneration of the hydrogen spectrum, while in the proof by Schrödinger this spectrum has still accidental degeneration. The hidden conserved Lenz vector is what is missing in the treatment by Schrödinger.

We consider Pauli's proof below and the proof above of Kepler's law of ellipses as proofs from "The Book". This is an expression of the remarkable mathematician Paul Erdös, who supposed that God has a book in which he keeps only the most beautiful proofs of mathematical theorems. One of the joyful parts of mathematics is to strive for proofs from "The Book".

We shall only outline the method of Pauli, and leave the proofs of the formulas to the interested reader. In quantum mechanics the three components of the position $\mathbf{r} = (r_1, r_2, r_3)$ and the momentum $\mathbf{p} = (p_1, p_2, p_3)$ are selfadjoint operators on a Hilbert space $\mathcal{H}$, satisfying the Heisenberg commutation relations

$$[r_i, r_j] = [p_i, p_j] = 0 \ , \ [p_i, r_j] = -i\hbar\delta_{ij} \ .$$

We wish to quantize the energy H, the angular momentum $\mathbf{L}$ and the Lenz vector $\mathbf{K}$. For H and $\mathbf{L}$ we can just take the classical formulas

$$H = p^2/2m - k/r \ , \ \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

but for the Lenz vector $\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r$ there is an ambiguity with the definition of $\mathbf{p}\times\mathbf{L}$. Indeed, for the first component should we take $p_2L_3 - p_3L_2$ or $L_3p_2 - L_2p_3$? For the solution of this ambiguity Pauli chose the average

$$\mathbf{K} = (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p})/2 - km\mathbf{r}/r \ ,$$

which is easily seen to be selfadjoint. The following formulas

$$[L_i, r_j] = i\hbar\epsilon_{ijk}r_k \ , \ [L_i, p_j] = i\hbar\epsilon_{ijk}p_k$$

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \ , \ [L_i, K_j] = i\hbar\epsilon_{ijk}K_k$$

mean that $\mathbf{r}, \mathbf{p}, \mathbf{L}$ and $\mathbf{K}$ are so called vector operators. Here ϵ_{ijk} is the totally antisymmetic ϵ-tensor, that is $\epsilon_{ijk} = 1$ if $ijk = 123, 231, 312$, $\epsilon_{ijk} = -1$ if $ijk = 213, 132, 321$ and zero otherwise. The conservation of $\mathbf{L}$ and $\mathbf{K}$ amounts to the relations

$$[H, \mathbf{L}] = [H, \mathbf{K}] = 0 \ .$$

The first relation is immediate because H has spherical symmetry, but the second relation requires some calculation. The following formulas are in perfect analogy with classical mechanics

$$\mathbf{L} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{L} = 0 \ ,$$

but the formula

$$K^2 = 2mH(L^2 + \hbar^2) + k^2m^2$$

has a "quantum correction" compared with the classical formula, vanishing in the limit for $\hbar \to 0$. Our last formula

$$[K_i, K_j] = i\hbar\epsilon_{ijk}(-2mH)L_k$$

is obtained after a careful calculation.

The sometimes cumbersome calculations behind the above formulas shall now be rewarded by an elegant and conceptually clear method for finding the hydrogen spectrum. We wish to compute the dimension of the eigenspace

$$\{\psi \in \mathcal{H}; H\psi = E\psi\}$$

with a fixed eigenvalue $E < 0$. If we denote

$$\mathbf{I} = (\mathbf{L} + (-2mE)^{-1/2}\mathbf{K})/2 \ , \ \mathbf{J} = (\mathbf{L} - (-2mE)^{-1/2}\mathbf{K})/2$$

then it is easy to see that $\mathbf{I}$ and $\mathbf{J}$ satisfy the commutation relations

$$[I_i, I_j] = i\hbar\epsilon_{ijk}I_k \ , \ [J_i, J_j] = i\hbar\epsilon_{ijk}J_k \ , \ [I_i, J_j] = 0 \ .$$

In other words the six dimensional vector space spanned by the components of $\mathbf{L}$ and $\mathbf{K}$ is a Lie algebra isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ commuting with H. The first $\mathfrak{sl}_2$ has basis I_i and the second $\mathfrak{sl}_2$ has basis J_j. From $\mathbf{L}^\star = \mathbf{L}, \mathbf{K}^\star = \mathbf{K}$ and $E < 0$ we deduce that $\mathbf{I}^\star = \mathbf{I}, \mathbf{J}^\star = \mathbf{J}$. In other words the real six dimensional vector space spanned by the components of $\mathbf{L}$ and $\mathbf{K}$ (or equivalently spanned by the components of $\mathbf{I}$ and $\mathbf{J}$) becomes identified with $i\mathfrak{su}_2 \oplus i\mathfrak{su}_2$.

Assuming that the spectrum has natural degeneration with respect to angular momentum and Lenz vector we can conclude

$$\{\psi \in \mathcal{H}; H\psi = E\psi\} = L(m) \otimes L(n)$$

for some $m, n \in \mathbb{N}$. Here I_i and J_j work in the first and the second factor of $L(m) \otimes L(n)$ respectively. Because

$$2(I^2 - J^2) = (\mathbf{I} + \mathbf{J}) \cdot (\mathbf{I} - \mathbf{J}) + (\mathbf{I} - \mathbf{J}) \cdot (\mathbf{I} + \mathbf{J}) =$$

$$(-2mE)^{-1/2}(\mathbf{L} \cdot \mathbf{K} + \mathbf{K} \cdot \mathbf{L}) = 0 \ ,$$

and by Exercise 9.5 and Exercise 10.2 below

$$I^2|_{L(m)\otimes L(n)} = m(m+2)\hbar^2/4 \ , \ J^2|_{L(m)\otimes L(n)} = n(n+2)\hbar^2/4$$

we conclude that $m = n$ and

$$\{\psi \in \mathcal{H}; H\psi = E\psi\} = L(n) \otimes L(n)$$

for some $n \in \mathbb{N}$. Finally we shall derive a formula for E_n as function of n. Rewriting the above formula $K^2 = 2mH(L^2 + \hbar^2) + k^2m^2$ in the form

$$L^2 + (-2mE)^{-1}K^2 + \hbar^2 = -k^2m/2E$$

and because on the space $L(n) \otimes L(n)$

$$L^2 + (-2mE)^{-1}K^2 + \hbar^2 = 2(I^2 + J^2) + \hbar^2 = n(n+2)\hbar^2 + \hbar^2 = (n+1)^2\hbar^2$$

we arrive at

$$E = E_n = -k^2m/2(n+1)^2\hbar^2$$

with n running over the set $\mathbb{N}$. The energy level E_n has multiplicity $(n+1)^2$. As a representation for $\mathbf{I}$ and $\mathbf{J}$ it is irreducible of the form $L(n) \otimes L(n)$, but restricting to the diagonal subalgebra $\mathbf{L}$ we have the Clebsch-Gordan rule

$$L(n) \otimes L(n) = L(0) \oplus L(2) \oplus \cdots \oplus L(2n) \; .$$

If we forget the symmetry of the Lenz vector $\mathbf{K}$ and take into account only the spherical symmetry of the angular momentum vector $\mathbf{L} = \mathbf{I} + \mathbf{J}$ the n^{th} energy level has accidental degeneration of multiplicity $(n+1)$.

Exercise 10.1. *Show that for A, B selfadjoint operators on a Hilbert space $\mathcal{H}$ the operator $(AB+BA)/2$ is again selfadjoint.*

Exercise 10.2. *The Pauli spin matrices are defined by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ,$$

so $H = \sigma_3, E = (\sigma_1 + i\sigma_2)/2, F = (\sigma_1 - i\sigma_2)/2$ gives the relation between the Chevalley basis and the Pauli matrices. In a unitary representation of $\mathfrak{su}_2$ the Pauli matrices act as selfadjoint operators. Check the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

and show that the Casimir operator $C = H^2 + 2EF + 2FE$ is equal to $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$. The conclusion is that $L^2 = \hbar^2C/4$, so apart from a factor $\hbar^2/4$ the Casimir operator C is just the square length of the the angular momentum vector.

Exercise 10.3. *Show that the Cartesian space $\mathbb{R}^3$ with Lie bracket $[\cdot,\cdot]$ equal to the vector product $\times$ is a Lie algebra. Show that this Lie algebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{so}_3$. Show that $\mathfrak{g}_0$ has a basis E_i with $[E_i,E_j]=\epsilon_{ijk}E_k$.*

Exercise 10.4. *Suppose we are in the regime where $H=E>0$. Show that $\mathbf{L}^\star=\mathbf{L},\mathbf{K}^\star=\mathbf{K}$ implies that $\mathbf{I}^\star=\mathbf{J},\mathbf{J}^\star=\mathbf{I}$. Hence the Casimir operator I^2+J^2 acts on the eigenspace*

$$\{\psi\in\mathcal{H};H\psi=E\psi\}$$

as a selfadjoint operator. If we assume this eigenspace to be an irreducible representation of $\mathfrak{sl}_2\oplus\mathfrak{sl}_2$ then the operator I^2+J^2 acts by multiplication with a real number $s(s+2)\hbar^2/2$ by Schur's Lemma with $s\in-1+i\mathbb{R}$ or $s\in\mathbb{R}$. Because $E=-k^2m/2(s+1)^2\hbar^2>0$ we have $s\in-1+i\mathbb{R}$. The parametrization of the nonpositive real numbers by $(s+1)^2\hbar^2/2$ with $s\in-1+i\mathbb{R}$ will become clear in a later section on representations of the Lorentz algebra $\mathfrak{so}_{3,1}(\mathbb{R})$.

Exercise 10.5. *Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras. Suppose V_1 and V_2 are representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. We define the outer tensor product representation $V_1\boxtimes V_2$ of the direct sum $\mathfrak{g}_1\oplus\mathfrak{g}_2$ as follows: As a vector space $V_1\boxtimes V_2=V_1\otimes V_2$ and the representation is defined by*

$$(X_1,X_2)(v_1\otimes v_2)=(X_1v_1)\otimes v_2+v_1\otimes(X_2v_2)$$

for all $x_1\in\mathfrak{g}_1$, $x_2\in\mathfrak{g}_2$, $v_1\in V_1$ and $v_2\in V_2$. Show that the outer tensor product is indeed a representation of the direct sum $\mathfrak{g}_1\oplus\mathfrak{g}_2$. Show that the outer tensor product of two irreducible representations is again irreducible, and each irredicible representation of the direct sum $\mathfrak{g}_1\oplus\mathfrak{g}_2$ arises this way.

11 Spinning elementary particles

One can imagine an elementary particle as a small spherically symmetric ball, so with internal symmetry Lie algebra $\mathfrak{so}_3 \cong \mathfrak{sl}_2$. This internal symmetry can be quantized by the unitary irreducible representation $L(2s)$ of dimension $(2s+1)$ and highest weight $2s \in \mathbb{N}$. The parameter $s \in \mathbb{N}/2$ is called the spin J of the irreducible representation $L(2s)$ of $\mathfrak{so}_3 \cong \mathfrak{sl}_2$. The faster the elementary particle spins the larger the spin quantum number $s \in \mathbb{N}/2$. Hence an elementary particle with spin $J = s \in \mathbb{N}/2$ can occupy $(2s+1)$ states indexed by a quantum number J_3 taken from the set $\{s, s-1, \cdots, -s\}$, which is just the set of eigenvalues of the Lie algebra element $h/2$ from $\mathfrak{sl}_2$. So the spin J is an index for the irreducible representation of $\mathfrak{sl}_2$, while the spin around the (third) axis J_3 is an index for a basis in this representation space. These notations are fairly standard.

Examples are the Higgs particle with spin 0, the electron with spin 1/2, the photon with spin 1 or the graviton with spin 2. For example, the electron with spin 1/2 has two quantum numbers, taken from the set $\{1/2, -1/2\}$. Equivalently an electron has spin up $(u \leftrightarrow 1/2)$ or spin down $(d \leftrightarrow -1/2)$. The notion of electron spin was proposed by Goudsmit and Uhlenbeck in 1926, at a time they were still graduate students under Paul Ehrenfest in Leiden. There are no elementary particles known to exist with spin greater than 2. Apparently if an elementary particle spins too fast it becomes unstable, preventing its existence.

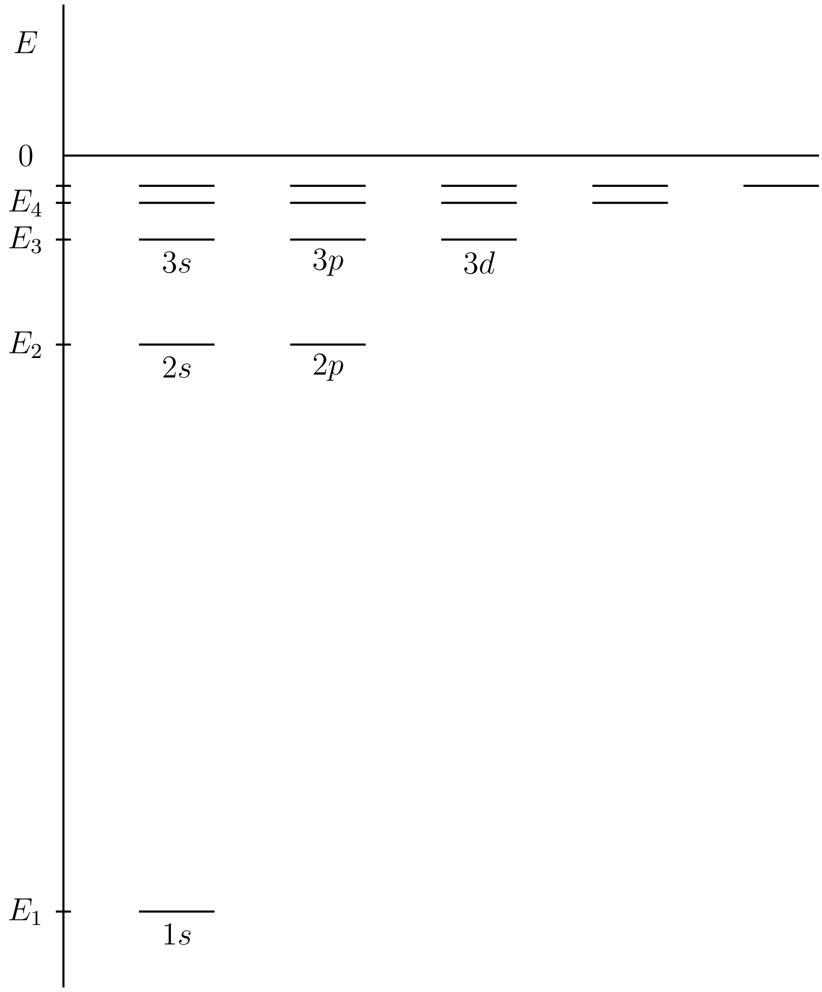
Particles with integral spin are called bosons, and particles with half integral spin are called fermions. Suppose an elementary particle with spin s has as state space a Hilbert space $\mathcal{H}$. The "spin statistics theorem" states that a system of n such identical particles has as state space the n^{th} symmetric power $S^n(\mathcal{H})$ or the n^{th} antisymmetric power $A^n(\mathcal{H})$, depending on whether the particle has integral spin or half integral spin respectively. An identical system of n bosons can live happily together in a same state $\psi^n \in S^n(\mathcal{H})$, but for an identical system of n fermions no two particles can live together in the same state, because $\psi \wedge \psi \wedge \psi_3 \wedge \cdots \wedge \psi_n$ vanishes in $A^n(\mathcal{H})$. This is the famous Pauli exclusion principle: two electrons can never occupy the same state.

The spectrum of the Kepler problem has been computed before using the angular momentum vector $\mathbf{L}$ and the Lenz vector $\mathbf{K}$, which together generate for negative energy $E < 0$ a Lie algebra $\mathfrak{so}_4$. The discrete energy spectrum is located at $E_{n+1} = -1/(n+1)^2$ for $n \in \mathbb{N}$ in suitable units. The $(n+1)^{st}$

energy level is degenerated with multiplicity $(n+1)^2$, and transforms under $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as the irreducible representation $L(n) \otimes L(n)$. The $\mathfrak{so}_3 \cong \mathfrak{sl}_2$ corresponding to the angular momentum vector $\mathbf{L}$ lies inside $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as the diagonal, and the restriction of $L(n) \otimes L(n)$ to this diagonal is given by the Clebsch-Gordan rule

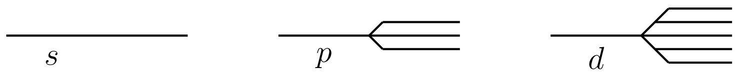
$$L(n) \otimes L(n) = L(0) \oplus L(2) \oplus \cdots \oplus L(2n) \ .$$

Therefore the spectrum of the Kepler problem can be pictorially described by the following figure.



The shells indicated with an s correspond to the one dimensional irreducible representation $L(0)$. The shells indicated with a p correspond to the three dimensional irreducible representation $L(2)$. The shells indicated with a d correspond to the five dimensional irreducible representation $L(4)$. The shells indicated with an f correspond to the seven dimensional irreducible representation $L(6)$, and so on. The elements of the periodic system can be understood to consist of a nucleus (consisting of protons and neutrons) surrounded by a cloud of electrons. Let us assume for simplicity that the electrons only interact with the nucleus, so the mutual interaction among the electrons is neglected. Then the energy levels of our Kepler problem are filled with electrons according to Pauli's exclusion principle from the bottom up, and for fixed energy first the shell s, subsequently the shell p, then the shell d, and so on. For example, the hydrogen atom H has one electron in the shell $1s$, leaving one open place in shell $1s$. The carbon atom C occurs in the periodic system on place number 6. The shells $1s, 2s$ are completely filled with $2+2=4$ electrons, and the shell $2p$ is occupied with 2 electrons, leaving four open places in shell $2p$. This is the reason for the existence of the chemical binding CH_4.

In the situation of a constant electric or magnetic field along some axis the energy levels will split up according to their degeneration, so



This splitting of spectral lines is called the Stark-Zeeman effect. It is a consequence of symmetry breaking from $\mathfrak{so}_3 \cong \mathfrak{sl}_2$ to $\mathfrak{gl}_1$.